

A threshold result for loose Hamiltonicity in random regular uniform hypergraphs

Daniel Altman Catherine Greenhill*

School of Mathematics and Statistics
UNSW Australia
Sydney, NSW 2052, Australia

daniel.h.altman@gmail.com c.greenhill@unsw.edu.au

Mikhail Isaev*

School of Mathematics and Statistics
UNSW Australia
Sydney, NSW 2052, Australia

Moscow Institute of Physics and Technology
Dolgoprudny, 141700, Russia

isaev.m.i@gmail.com

Reshma Ramadurai

Department of Mathematics
The University of Waikato
Hamilton 3240, New Zealand

reshmar@waikato.ac.nz

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Abstract

Let $\mathcal{G}(n, r, s)$ denote the uniform model of random r -regular s -uniform hypergraphs on n vertices, where s is a fixed constant and $r = r(n)$ may grow with n . An ℓ -overlapping Hamilton cycle is a Hamilton cycle in which successive edges overlap in precisely ℓ vertices, and 1-overlapping Hamilton cycles are called *loose* Hamilton cycles.

When $r, s \geq 3$ are fixed integers, we establish a threshold result for the property of containing a loose Hamilton cycle. This partially verifies a conjecture of Dudek, Frieze, Ruciński and Šileikis (2015). In this setting, we also find the asymptotic distribution and expected value of the number of loose Hamilton cycles in $\mathcal{G}(n, r, s)$.

Finally we prove that for $\ell = 2, \dots, s - 1$ and for r growing moderately as $n \rightarrow \infty$, the probability that $\mathcal{G}(n, r, s)$ has a ℓ -overlapping Hamilton cycle tends to zero.

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1 Introduction

A *hypergraph* $G = (V, E)$ consists of a finite set V of vertices and a multiset E of multisubsets of V , which we call edges. We say that H is *simple* if E is a set of sets: that is, there are no repeated edges and no edge contains a repeated vertex. Given a fixed integer $s \geq 2$, the hypergraph G is said to be *s-uniform* if every edge contains precisely s vertices, counting multiplicities. Uniform hypergraphs have been well-studied, as they generalise graphs (which are 2-uniform hypergraphs). Let $r \geq 1$ be an integer. A hypergraph is said to be *r-regular* if every vertex has degree r , counting multiplicities. (For more background on hypergraphs, see [3].)

For integers $r, s \geq 2$, let $\mathcal{G}(n, r, s)$ be the uniform probability space on the set $\mathcal{S}(n, r, s)$ of all simple, r -regular, s -uniform hypergraphs on the vertex set $\{1, 2, \dots, n\}$. To avoid trivialities, assume that s divides rn (as any hypergraph in $\mathcal{S}(n, r, s)$ has rn/s edges). We write $G \in \mathcal{G}(n, r, s)$ to denote that G is a hypergraph chosen uniformly from $\mathcal{S}(n, r, s)$. Here s is fixed, though we sometimes allow $r = r(n)$ to grow with n .

There are many ways to define cycles in hypergraphs. Most generally, for $k \geq 2$, a *k-cycle* is a set of k edges which can be labelled as e_0, e_1, \dots, e_{k-1} such that $e_j \cap e_{j+1}$ is nonempty for $j = 0, \dots, k-1$ (indices taken modulo k). A *1-cycle* is an edge which contains a repeated vertex. Hence a hypergraph is simple if and only if it contains no 1-cycle and no 2-cycle consisting of two identical edges.

For a fixed integer $k \geq 2$, a *loose k-cycle* is a sequence of edges $(e_0, e_1, \dots, e_{k-1})$ such that

$$|e_i \cap e_j| = \begin{cases} 1 & \text{if } |i - j| = 1 \pmod k, \\ 0 & \text{otherwise.} \end{cases}$$

A loose k -cycle C contains precisely $k(s-1)$ vertices. Figure 1 shows a loose 6-cycle in a 3-uniform hypergraph (the other edges of the hypergraph are not shown).

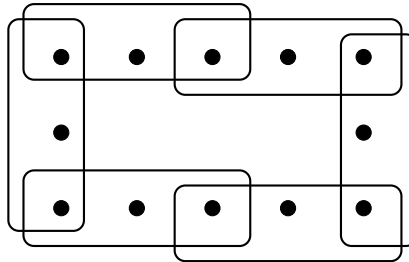


Figure 1: A loose 6-cycle in a 3-uniform hypergraph.

A loose 1-cycle is a 1-cycle which contains exactly $s-1$ distinct vertices; that is, it contains one vertex v with multiplicity 2, and the remaining $s-2$ vertices are distinct from v and from each other.

Throughout the paper, asymptotics are as $n \rightarrow \infty$, restricted to those n such that $s - 1$ divides n and s divides rn . If the probability of an event tends to 1 as $n \rightarrow \infty$ then we say that the event holds *asymptotically almost surely* (a.a.s.).

Define $t = n/(s - 1)$ and let G be a hypergraph on n vertices. Observe that a loose t -cycle of G covers all $(s - 1)t = n$ vertices of G . From now on, we refer to a loose t -cycle of G as a *loose Hamilton cycle*. A necessary condition for the existence of a loose Hamilton cycle in a hypergraph $G \in \mathcal{S}(n, r, s)$ is that $s - 1$ divides n .

More generally, an ℓ -overlapping Hamilton cycle is a set of $t_\ell = n/(s - \ell)$ edges which can be labelled $e_0, e_1, \dots, e_{t_\ell-1}$ such that for some ordering v_0, \dots, v_{n-1} of the vertices we have

$$e_i = \{v_{i(s-\ell)}, v_{i(s-\ell)+1}, \dots, v_{i(s-\ell)+s-1}\} \quad \text{for } i = 0, \dots, t_\ell - 1. \quad (1.1)$$

Here the vertex labels are also interpreted cyclically, so that for example

$$e_{t_\ell-1} = \{v_{n-s+\ell}, v_{n-s+\ell+1}, \dots, v_{n-1}, v_0, v_1, \dots, v_{\ell-1}\}.$$

A necessary condition for an ℓ -overlapping Hamilton cycle to exist in an s -uniform hypergraph on n vertices is that $s - \ell$ divides n . An $(s - 1)$ -overlapping Hamilton cycle is also called a *tight* Hamilton cycle, and a 1-overlapping Hamilton cycle is just a loose Hamilton cycle.

The case $s = 2$ (graphs) has been extensively studied. In order to prove that random r -regular graphs are a.a.s. Hamiltonian, Robinson and Wormald [12, 13] used an analysis of variance technique now known as the *small subgraph conditioning method*. The calculations of Robinson and Wormald [12] provide the asymptotic distribution of the number of Hamilton cycles for cubic graphs ($r = 3$), but their generalisation [13] to higher degrees used an inductive argument based on the a.a.s. presence of perfect matchings in random regular graphs of degree $r \geq 4$. Frieze et al. [8] made a direct investigation of the number of Hamilton cycles in random regular graphs, with an algorithmic perspective. Janson [11] observed that the calculations of Frieze et al. [8] could be adapted to give the asymptotic distribution of the number of Hamilton cycles in random r -regular graphs. Janson stated this distribution in [11, Theorem 2], and noted that in particular, the expected number of Hamilton cycles in random r -regular graphs is asymptotically equal to

$$e \sqrt{\frac{\pi}{2n}} \left(\frac{(r - 2)^{(r-2)/2} (r - 1)}{r^{(r-2)/2}} \right)^n. \quad (1.2)$$

Janson also observed that [11, Theorem 2] implies that random r -regular graphs are a.a.s. Hamiltonian, when $r \geq 3$.

Our aim in this paper is to extend the results of Frieze et al. [8] and Janson [11] by using small subgraph conditioning to study loose Hamilton cycles in random s -uniform r -regular hypergraphs, for any $r, s \geq 2$. Our work is also motivated by two conjectures stated by Dudek et al [2], as discussed in Section 1.1. Where possible, we state our results so that they also cover the known results for graphs ($s = 2$).

Theorem 1.1. *Let $s \geq 2$ be a fixed integer. There exists a positive constant $\rho(s)$ such that for any integer $r \geq 2$,*

$$\lim_{n \rightarrow \infty} \Pr(G \in \mathcal{G}(n, r, s) \text{ contains a loose Hamilton cycle}) \rightarrow \begin{cases} 1 & \text{if } r > \rho(s), \\ 0 & \text{if } r \leq \rho(s). \end{cases}$$

Specifically, $\rho = \rho(s)$ is the unique real number in $[2, \infty)$ such that

$$(\rho - 1)(s - 1) \left(\frac{\rho s - \rho - s}{\rho s - \rho} \right)^{(s-1)(\rho s - \rho - s)/s} = 1.$$

We note that $\rho(2) \in (2, 3)$ and $\rho(3) = 3$, while if $s \geq 4$ then

$$\rho^-(s) < \rho(s) < \rho^+(s) \tag{1.3}$$

where

$$\begin{aligned} \rho^-(s) &= \frac{e^{s-1}}{s-1} - \frac{s-2}{2} - \frac{(s^2 - s + 1)^2}{se^{s-1}}, \\ \rho^+(s) &= \frac{e^{s-1}}{s-1} - \frac{s-2}{2}. \end{aligned}$$

In Table 1 we give the values of $\rho(s)$ for $s = 2, \dots, 10$, and for $s = 4, \dots, 10$ we compare $\rho(s)$ with the lower and upper bounds $\rho^-(s)$, $\rho^+(s)$ given in (1.3). All values are rounded to three decimal places, except $\rho(3) = 3$, since it is an integer. (Three decimal places are required to see that $\rho(5) < 12$.) We see that $\rho(s)$ is closely approximated by the upper bound $\rho^+(s)$, except at very small values of s .

s	2	3	4	5	6	7	8	9	10
$\rho^-(s)$	–	–	2.891	10.130	26.388	63.974	153.239	368.896	896.229
$\rho(s)$	2.488	3	5.501	11.998	27.580	64.675	153.625	369.100	896.332
$\rho^+(s)$	–	–	5.695	12.150	27.683	64.738	153.662	369.120	896.342

Table 1: Values of $\rho(s)$ for small s , together with our bounds (for $s \geq 4$).

Let Y_G be the number of loose Hamilton cycles in $G \in \mathcal{G}(n, r, s)$. When $s \geq 3$ and $r > \rho(s)$, our calculations provide the asymptotic distribution of Y_G , stated later as Theorem 6.2. Our calculations also lead to an asymptotic expression for the expected value of Y_G , for any fixed $r, s \geq 2$ with $(r, s) \neq (2, 2)$.

Theorem 1.2. *Suppose that $r, s \geq 2$ are fixed integers with $(r, s) \neq (2, 2)$. Then*

$$\mathbb{E}(Y_{\mathcal{G}}) \sim \exp\left(\frac{(s-1)(rs-s-2)}{2(rs-r-s)}\right) \sqrt{\frac{\pi}{2n}} (s-1) \times \left((r-1)(s-1) \left(\frac{rs-r-s}{rs-r}\right)^{(s-1)(rs-r-s)/s}\right)^{n/(s-1)}.$$

Observe that this expectation tends to zero whenever $r \leq \rho(s)$. In particular, this implies the negative side of our threshold result, Theorem 1.1.

When $s = 2$ and $r \geq 3$, Theorem 1.2 matches the expression proved by Janson [11, Theorem 2], quoted above in (1.2). When $r = s = 2$, a 2-regular graph is Hamiltonian if and only if it is connected, and a Hamiltonian 2-regular graph contains exactly one Hamilton cycle. Hence in this case, $\mathbb{E}(Y_{\mathcal{G}})$ equals the probability that a random 2-regular graph is Hamiltonian, and Wormald [14, Equation (11)] noted that this probability is asymptotically equal to

$$\frac{e^{3/4}}{2} \sqrt{\frac{\pi}{n}}.$$

Our final result concerns ℓ -overlapping Hamilton cycles and allows the degree r to grow moderately with n .

Theorem 1.3. *Let $s \geq 3$ be a fixed integer and let*

$$\kappa = \kappa(s) = \begin{cases} 1 & \text{if } s \geq 4, \\ \frac{1}{2} & \text{if } s = 3. \end{cases}$$

Suppose that $r = r(n)$ with $3 \leq r = o(n^{\kappa})$. Then a.a.s. $\mathcal{G}(n, r, s)$ has no ℓ -overlapping Hamilton cycle for $\ell = 2, \dots, s-1$.

To prove these results, as is usual in this area, we will work in a related probability model known as the *configuration model*. After discussing some related results and extensions in Section 1.1, we review the configuration model for hypergraphs in Section 2 and prove Theorem 1.3 in Section 2.1. To prove our remaining results we will apply the small subgraph conditioning method, which is discussed in Section 2.2. The structure of the rest of the paper will be described in Section 2.3.

1.1 Extensions and related results

The small subgraph conditioning method has been applied to prove many a.a.s. structural theorems (contiguity results) for regular graphs. See Wormald [14] or Janson [11] for more detail. For uniform regular hypergraphs, we only know of one application of the method: Cooper et al. [2] used small subgraph conditioning to investigate perfect matchings in random regular uniform hypergraphs. They proved a threshold result for

existence of a perfect matching in a random r -regular s -uniform hypergraph, where $r, s \geq 2$ are fixed integers. Specifically, in [2, Theorem 1] they proved that as $n \rightarrow \infty$, this probability tends to 0 if $s > \sigma_r$ and tends to 1 if $s < \sigma_r$, where

$$\sigma_r = \frac{\ln r}{(r-1) \ln \left(\frac{r}{r-1} \right)} + 1.$$

Defining $r_0(s) = \min\{r : s < \sigma_r\}$, Cooper et al. [2] remark that $r_0(s)$ is approximately e^{s-1} . It is interesting to observe that, in contrast to graphs ($s = 2$), the threshold for the existence of perfect matchings in $\mathcal{G}(n, r, s)$ is higher than the threshold for the existence of loose Hamilton cycles when $s \geq 3$; that is, $r_0(s) > \rho(s)$.

Recently, Dudek et al. [6] established a relation between $\mathcal{G}(n, r, s)$ and the uniform probability space on the set of s -uniform hypergraphs on n vertices with m edges (when $s \geq 3$). Using known results about the existence of loose Hamilton cycles in the latter model, they showed that a.a.s. $G \in \mathcal{G}(n, r, s)$ contains a loose Hamilton cycle when $r \gg \ln n$ (or $r = \Omega(\ln n)$, if $s = 3$) and $r = o(n^{1/2})$.

Dudek et al. made the following conjecture [6, Conjecture 1], rewritten here in our notation:

For every $s \geq 3$ there exists a constant $\rho = \rho(s)$ such that for any $r \geq \rho$, $\mathcal{G}(n, r, s)$ contains a loose Hamilton cycle a.a.s.

We have partially verified this conjecture with Theorem 1.1, which gives a threshold result for constant values of r . This leaves a gap for degrees $r = r(n) = O(\ln n)$. Intuitively, it seems that increasing the degree should make the existence of a loose Hamilton cycle more likely. However, as the small subgraph conditioning method does not apply directly when r is growing, it appears that other ideas are required, such as a switching argument.

There has been much work on ℓ -overlapping Hamilton cycles in the binomial model $\mathcal{G}_{n,p}^{(s)}$ of s -uniform hypergraphs, where each s -set is an edge with probability p , independently. In particular, loose ($\ell = 1$) and tight ($\ell = s - 1$) Hamilton cycles are well studied, see for example [1, 4, 7] and references therein. Dudek and Frieze [4, Theorem 3(i)] proved that for all fixed integers $s > \ell \geq 2$ and fixed $\epsilon > 0$, if $p \leq (1 - \epsilon)e^{s-\ell}/n^{s-\ell}$ then a.a.s. $\mathcal{G}_{n,p}^{(s)}$ has no ℓ -overlapping Hamilton cycle. This motivated the second conjecture of Dudek et al. [6, Conjecture 2], written here in our notation:

For every $s > \ell \geq 2$, if $r \gg n^{\ell-1}$ then a.a.s. $\mathcal{G}(n, r, s)$ contains an ℓ -overlapping Hamilton cycle.

Theorem 1.3 can be seen as a contribution towards the proof of the corresponding negative result needed to establish a degree threshold for the existence of an ℓ -overlapping Hamilton cycle. We believe that complex-analytic methods such as those presented in [10] may allow progress towards the proof of [6, Conjecture 2], and will investigate this in future work.

2 Main ideas

We work in a natural generalisation of the configuration model to hypergraphs. This is the same model used by Cooper et al. [2]. We use the notation $[n] = \{1, 2, \dots, n\}$.

Let B_1, B_2, \dots, B_n be disjoint sets of size r , which we call *cells*, and define $\mathcal{B} = \cup_{i=1}^n B_i$. Elements of \mathcal{B} are called *points*. We assume that there is a fixed ordering on the rn points of \mathcal{B} (so that different points in the same cell are distinguishable).

Assume that s divides rn and $s-1$ divides n . Let $\Omega(n, r, s)$ be the set of all unordered partitions $F = \{U_1, \dots, U_{rn/s}\}$ of \mathcal{B} into rn/s parts, where each part has exactly s points. Each partition $F \in \Omega(n, r, s)$ defines a hypergraph $G(F)$ on the vertex set $[n]$ in a natural way: vertex i corresponds to the cell B_i , and each part $U \in F$ gives rise to an edge e_U such that the multiplicity of vertex i in e_U equals $|U \cap B_i|$, for $i = 1, \dots, n$. Then $G(F)$ is an s -uniform r -regular hypergraph. The partition $F \in \Omega(n, r, s)$ is called *simple* if $G(F)$ is simple. More generally, we will often describe $F \in \Omega(n, r, s)$ as having a particular hypergraph property if $G(F)$ has that property.

For any positive integer ℓ which is divisible by s , define

$$p(\ell) = \frac{\ell!}{(\ell/s)! (s!)^{\ell/s}}. \quad (2.1)$$

Then

$$|\Omega(n, r, s)| = p(rn) = \frac{(rn)!}{(rn/s)! (s!)^{rn/s}}. \quad (2.2)$$

Now let $\mathcal{F}(n, r, s)$ be the uniform probability space on $\Omega(n, r, s)$, where each partition is assigned probability $1/|\Omega(n, r, s)|$. Again, we write $F \in \mathcal{F}(n, r, s)$ to denote that F is a partition chosen uniformly from $\Omega(n, r, s)$. Every hypergraph in $\mathcal{G}(n, r, s)$ corresponds to precisely $(r!)^n$ partitions $F \in \Omega(n, r, s)$. (This is only true for simple hypergraphs.) Therefore

$$|\mathcal{G}(n, r, s)| = \frac{|\Omega(n, r, s)| \Pr(F \in \mathcal{F}(n, r, s) \text{ is simple})}{(r!)^n}. \quad (2.3)$$

It was shown in [2] that when $r, s \geq 3$ are fixed,

$$\lim_{n \rightarrow \infty} \Pr(F \in \mathcal{F}(n, r, s) \text{ is simple}) = e^{-(r-1)(s-1)/2}. \quad (2.4)$$

Now let \mathcal{A} be any event in $\Omega(n, r, s)$ with probability $o(1)$. Then for a random partition $F \in \mathcal{F}(n, r, s)$,

$$\Pr(\mathcal{A} \mid F \text{ is simple}) \leq \frac{\Pr(\mathcal{A})}{\Pr(F \text{ is simple})} = o(1), \quad (2.5)$$

which implies that the corresponding event in $\mathcal{G}(n, r, s)$ also has probability $o(1)$.

2.1 Expected value in configuration model

For $\ell = 1, \dots, s-1$, let $Y^{(\ell)}$ be the number of ℓ -overlapping Hamilton cycles in $\mathcal{F}(n, r, s)$. We now calculate the expected value of $Y^{(\ell)}$.

Lemma 2.1. *Let $\ell \in \{1, \dots, s-1\}$, where $s \geq 3$ is fixed, and let $r = r(n)$ be a function of n which satisfies $r \geq 3$. Write $\ell = q(s-\ell) + c$ where q, c are nonnegative integers and $c \in \{0, 1, \dots, s-\ell-1\}$. Then*

$$\mathbb{E}Y^{(\ell)} \sim \sqrt{\frac{\pi}{2n}} (s-\ell) \left(\frac{e}{n}\right)^{(\ell-1)n/(s-\ell)} \times \left(\frac{((r)_{q+1})^{s-\ell} (r-q-1)^c (s-1)_{c+\ell-1} (rs-r\ell-s)^{(s-1)(rs-r\ell-s)/s}}{c! r^{r(s-1)(s-\ell)/s} (s-\ell)^{(s-1)(rs-r\ell-s)/s}} \right)^{n/(s-\ell)}.$$

Proof. Recall the notation $t_\ell = n/(s-\ell)$. There are $n!$ ways to fix an ordering v_0, \dots, v_{n-1} of the vertices. This gives rise to an ℓ -overlapping Hamilton cycle H with edges $e_0, \dots, e_{t_\ell-1}$ defined by (1.1). For $j = 0, \dots, t_\ell-1$ define the set $S_j = e_j \setminus e_{j-1}$, with index arithmetic performed cyclically (so $S_0 = e_0 \setminus e_{t_\ell-1}$). The sets S_0, \dots, S_{t_ℓ} partition $[n]$.

First, observe that $s-\ell-c$ vertices in S_i have degree $q+1$ in the Hamilton cycle H , and the remaining c vertices of S_i have degree $q+2$ in H , for all $i \in \{0, 1, \dots, t_\ell-1\}$. It follows that the chosen Hamilton cycle H (as a hypergraph) corresponds to precisely

$$2t_\ell (c! (s-\ell-c)!)^{t_\ell}$$

orderings of the vertices.

The number of ways to embed the edges of H as parts in a partition F is

$$\left(((r)_{q+1})^{s-\ell-c} ((r)_{q+2})^c \right)^{t_\ell} = \left(((r)_{q+1})^{s-\ell} (r-q-1)^c \right)^{t_\ell},$$

This completely specifies the t_ℓ parts of the partition which correspond to H . Finally, we must multiply by

$$\frac{p(rn - st_\ell)}{p(rn)}$$

for the probability that a randomly chosen partition contains these t_ℓ specified parts. We have shown that

$$\begin{aligned} \mathbb{E}Y^{(\ell)} &= \frac{n!}{2t_\ell (c! (s-\ell-c)!)^{t_\ell}} \left(((r)_{q+1})^{s-\ell} (r-q-1)^c \right)^{t_\ell} \frac{p(rn - st_\ell)}{p(rn)} \\ &= \frac{n!}{2t_\ell (c! (s-\ell-c)!)^{t_\ell}} \left(((r)_{q+1})^{s-\ell} (r-q-1)^c \right)^{t_\ell} \frac{(rn - st_\ell)! (rn/s)! (s!)^{rn/s}}{(rn/s - t_\ell)! (s!)^{rn/s - t_\ell} (rn)!}, \end{aligned}$$

and the result follows by applying Stirling's formula. \square

At this point we can also prove Theorem 1.3.

Proof of Theorem 1.3. Recall from the statement of Theorem 1.3 that $\kappa = \kappa(s)$ equals 1 if $s \geq 4$, and equals $\frac{1}{2}$ when $s = 3$. Fix $\ell = 2, \dots, s-1$, where $s \geq 3$ is constant and $r = r(n)$ may grow with n , such that $3 \leq r = o(n^\kappa)$. It follows from Lemma 2.1 that

$$\mathbb{E}Y^{(\ell)} \leq \left(\frac{c' r}{n^{\ell-1}} \right)^{n/(s-\ell)}$$

for some positive constant c' . Dudek et al. [5, Theorem 1] proved that when $r = o(n^\kappa)$,

$$|\mathcal{G}(n, r, s)| = \frac{(rn)!}{(rn/s)!(s!)^{rn/s}(r!)^n} \exp \left(-\frac{1}{2}(r-1)(s-1) + O((r/n)^{1/2} + r^2/n) \right).$$

Combining this with (2.2) and (2.3), we conclude that when $r = o(n^\kappa)$,

$$\Pr(F \in \mathcal{F}(n, r, s) \text{ is simple}) = \exp \left(-\frac{1}{2}(r-1)(s-1)(1+o(1)) \right) = \Omega(\exp(-\hat{c}r))$$

for some positive constant \hat{c} (independent of r). By (2.5) and Markov's Lemma, it follows that the probability that $G \in \mathcal{G}(n, r, s)$ contains an ℓ -overlapping Hamilton cycle is bounded above by

$$\begin{aligned} \frac{\mathbb{E}(Y^{(\ell)})}{\Pr(F \text{ is simple})} &\leq O(\exp(\hat{c}r)) \left(\frac{c' r}{n^{\ell-1}} \right)^{n/(s-\ell)} \\ &= O(1) \exp \left(\hat{c}r - \frac{n}{s-\ell} \ln \left(\frac{n^{\ell-1}}{c' r} \right) \right) \end{aligned}$$

which is $o(1)$ for any $r = o(n)$. This completes the proof. \square

We are particularly interested in loose Hamilton cycles ($\ell = 1$) for fixed r . Recall that $Y = Y^{(1)}$ is the number of loose Hamilton cycles in $\mathcal{F}(n, r, s)$.

Corollary 2.2. *Let $r, s \geq 3$ be fixed integers and let $t = n/(s-1)$. The expected value of Y satisfies*

$$\begin{aligned} \mathbb{E}(Y) &= \frac{n!}{2t((s-2)!)^t} \cdot r^n(r-1)^t \cdot \frac{p(rn-st)}{p(rn)} \\ &\sim \sqrt{\frac{\pi}{2n}} (s-1) \left((r-1)(s-1) \left(\frac{rs-r-s}{rs-r} \right)^{(s-1)(rs-r-s)/s} \right)^{n/(s-1)}. \end{aligned} \tag{2.6}$$

In Lemma 6.1 we characterise pairs (r, s) for which $\mathbb{E}(Y)$ tends to infinity, leading to the definition of the threshold function $\rho(s)$. Combining this result with (2.5), we obtain the negative part of the threshold result Theorem 1.1, as explained in Section 6. In order to complete the proof of Theorem 1.1, and prove Theorem 1.2, we require more information about the asymptotic distribution of the number of loose Hamilton cycles in $\mathcal{F}(n, r, s)$. This information is obtained using the small subgraph conditioning method.

2.2 Small subgraph conditioning for hypergraphs

The following statement of the small subgraph conditioning method is adapted from [11, Theorem 1]. A similar theorem is given in [14, Theorem 4.1].

Theorem 2.3 ([11]). *Let $\lambda_k > 0$ and $\delta_k \geq -1$, $k = 1, 2, \dots$, be constants and suppose that for each n there are random variables $X_{k,n}$, $k = 1, 2, \dots$, and Y_n (defined on the same probability space) such that $X_{k,n}$ is non-negative integer valued and $\mathbb{E}Y_n \neq 0$ and furthermore the following conditions are satisfied:*

(A1) $X_{k,n} \xrightarrow{d} Z_k$ as $n \rightarrow \infty$, jointly for all k , where $Z_k \sim \text{Po}(\lambda_k)$ are independent Poisson random variables;

(A2) For any finite sequence x_1, x_2, \dots, x_m of non-negative integers,

$$\frac{\mathbb{E}(Y_n | X_{1,n} = x_1, X_{2,n} = x_2, \dots, X_{m,n} = x_m)}{\mathbb{E}Y_n} \rightarrow \prod_{k=1}^m (1 + \delta_k)^{x_k} e^{-\lambda_k \delta_k} \quad \text{as } n \rightarrow \infty;$$

(A3) $\sum_{k \geq 1} \lambda_k \delta_k^2 < \infty$;

(A4) $\frac{\mathbb{E}Y_n^2}{(\mathbb{E}Y_n)^2} \rightarrow \exp\left(\sum_k \lambda_k \delta_k^2\right)$ as $n \rightarrow \infty$.

Then

$$\frac{Y_n}{\mathbb{E}Y_n} \xrightarrow{d} W = \prod_{k=1}^{\infty} (1 + \delta_k)^{Z_k} e^{-\lambda_k \delta_k} \quad \text{as } n \rightarrow \infty; \quad (2.7)$$

moreover, this and the convergence (A1) hold jointly. The infinite product defining W converges a.s. and in L^2 , with

$$\mathbb{E}W = 1 \text{ and } \mathbb{E}W^2 = \exp\left(\sum_{k \geq 1} \lambda_k \delta_k^2\right) = \lim_{n \rightarrow \infty} \frac{\mathbb{E}Y_n^2}{(\mathbb{E}Y_n)^2}.$$

Furthermore, if $\delta_k > -1$ for all k then a.a.s. $Y_n > 0$.

Recall that $t = n/(s-1)$ is the number of edges in a loose Hamilton cycle. We will apply Theorem 2.3 to the random variables defined as follows. In order to distinguish our specific random variables from the general random variables used in Theorem 2.3, we do not include the subscript n in our notation.

- Let Y be the number of subsets F_H of $F \in \mathcal{F}(n, r, s)$ consisting of t parts such that $G(F_H)$ is a loose Hamilton cycle in $G(F)$.

- For $k \geq 2$ let X_k be the number of subsets F_C of $F \in \mathcal{F}(n, r, s)$ consisting of k parts such that $G(F_C)$ is a loose k -cycle in $G(F)$.
- Let $X_1 = X_1$ be the number of parts U in $F \in \mathcal{F}(n, r, s)$ such that U gives rise to an edge which contains a repeated vertex. That is, X_1 is the number of parts U in $F \in \mathcal{F}(n, r, s)$ such that $|U \cap B_j| > 1$ for some $j \in [n]$.

Note that X_1 counts parts which correspond to 1-cycles in $G(F)$, not just loose 1-cycles. We define X_1 in this way so that $X_1 = 0$ if and only if no edge of $G(F)$ contains a repeated vertex. (Our definition of X_1 agrees with that used in [2].)

Cooper et al. proved in [2, Section 5] that $X_k \rightarrow Z_k$ as $n \rightarrow \infty$, jointly for $k \geq 1$, where $Z_k \sim \text{Po}(\lambda_k)$ are asymptotically independent Poisson random variables with mean integers k ,

$$\lambda_k = \frac{((r-1)(s-1))^k}{2k}. \quad (2.8)$$

This verifies that (A1) of Theorem 2.3 holds. In fact, Cooper et al. [2] worked with the random variable X'_k for $k \geq 1$ which counts the number of k -cycles (not necessarily loose). Here a sequence of k edges forms a k -cycle if successive pairs of edges overlap in at least one vertex (including the first and last edge of the sequence). Note that $X'_1 = X_1$. Calculations from [2, Section 5] show that $X'_k \stackrel{d}{\sim} X_k$ jointly for $k \geq 1$, since a.a.s. the contribution to X_k from non-loose k -cycles forms only a negligible fraction of X_k . Here we write $A_n \stackrel{d}{\sim} B_n$ to mean that two sequences of random variables (A_n) and (B_n) have the same asymptotic distribution, recalling that both X_k and X'_k depend on n . Hence Theorem 2.3 (A1) holds with λ_k as in (2.8).

In order to establish (A2) of Theorem 2.3, the following result (for general random variables) is convenient.

Lemma 2.4 ([11], Lemma 1). *Let $\lambda'_k \geq 0$, $k = 1, 2, \dots$, be constants. Suppose that (A1) holds, that $Y_n \geq 0$ and that*

(A2') *for every finite sequence x_1, x_2, \dots, x_m of non-negative integers*

$$\frac{\mathbb{E}(Y_n(X_{1,n})_{x_1}(X_{2,n})_{x_2} \cdots (X_{m,n})_{x_m})}{\mathbb{E}Y_n} \rightarrow \prod_{k=1}^m (\lambda'_k)^{x_k} \text{ as } n \rightarrow \infty.$$

Then (A2) holds with $\lambda_k(1 + \delta_k) = \lambda'_k$ for all $k \geq 1$.

It is routine to extend the arguments of Section 3.2 and Section 4 to show that

$$\frac{\mathbb{E}(Y(X_1)_{x_1}(X_2)_{x_2} \cdots (X_m)_{x_m})}{\mathbb{E}Y} = (1 + o(1)) \prod_{k=1}^m \left(\frac{\mathbb{E}(Y X_k)}{\mathbb{E}Y} \right)^{x_k}, \quad (2.9)$$

similarly to the case of Hamilton cycles in cubic graphs [12, equation (28)]. Roughly, (2.9) holds because fixing a constant-length cycle has asymptotically negligible effect on the number of ways to choose subsequent cycles, and noting that overlapping cycles also give negligible relative contribution. We omit these technical details, as is usual in the literature.

As Cooper et al. remark in [2, Section 2], the probability that two parts of $F \in \mathcal{F}(n, r, s)$ give rise to a repeated edge is $o(1)$; see also (4.6). Therefore the probability that $G(F)$ is simple is asymptotically equal to the probability that $Z_1 = 0$, which is $e^{-\lambda_1}$, as in (2.4). Furthermore, the random variable $Y_{\mathcal{G}}$ has the same asymptotic distribution as that of \widehat{Y} , where \widehat{Y} is the random variable obtained from Y by conditioning on the event $X_1 = 0$. In particular, $\mathbb{E}Y_{\mathcal{G}} \sim \mathbb{E}\widehat{Y} = \mathbb{E}(Y \mid X_1 = 0)$.

We now explain how the asymptotic distribution of $Y_{\mathcal{G}}$ (Theorem 6.2) can be obtained from the asymptotic distribution of Y , once the latter has been established using Theorem 2.3.

Lemma 2.5. *Suppose that Y and X_k satisfy conditions (A1)–(A4) of Theorem 2.3. Then*

$$\frac{Y_{\mathcal{G}}}{\mathbb{E}Y_{\mathcal{G}}} \xrightarrow{d} \prod_{k=2}^{\infty} (1 + \delta_k)^{Z_k} e^{-\lambda_k \delta_k} \quad \text{as } n \rightarrow \infty.$$

Moreover, if $\delta_k > -1$ for all $k \geq 1$ then a.a.s. $Y_{\mathcal{G}} > 0$.

Proof. Let \widehat{Y} be as defined above, and let \widehat{W} be the random variable obtained from W by conditioning on the event that $Z_1 = 0$. By Theorem 2.3 applied to Y , the convergence of (2.7) and the convergence of (A1) holds jointly. Therefore, after conditioning on the event $X_1 = 0$, which has non-vanishing probability, we have

$$\frac{\widehat{Y}}{\mathbb{E}\widehat{Y}} \xrightarrow{d} \widehat{W} \quad \text{and} \quad \frac{\mathbb{E}\widehat{Y}}{\mathbb{E}Y} \rightarrow \mathbb{E}\widehat{W}.$$

This implies that

$$\frac{Y_{\mathcal{G}}}{\mathbb{E}Y_{\mathcal{G}}} \stackrel{d}{\sim} \frac{\widehat{Y}}{\mathbb{E}\widehat{Y}} \cdot \frac{\mathbb{E}Y}{\mathbb{E}\widehat{Y}} \xrightarrow{d} \left(\lim_{n \rightarrow \infty} \frac{\mathbb{E}Y}{\mathbb{E}\widehat{Y}} \right) \widehat{W} = \frac{\widehat{W}}{\mathbb{E}\widehat{W}}.$$

Finally,

$$\widehat{W} = e^{-\lambda_1 \delta_1} \prod_{k=2}^{\infty} (1 + \delta_k)^{Z_k} e^{-\lambda_k \delta_k},$$

so $\mathbb{E}\widehat{W} = e^{-\lambda_1 \delta_1}$ and the first statement follows.

Next, suppose that $\delta_k > -1$ for all $k \geq 1$. Then a.a.s. $Y > 0$, by the final statement Theorem 2.3 applied to Y . The probability that F is simple is bounded below by a constant when $n \rightarrow \infty$, as observed by Cooper et al. [2]: see (2.4). Hence we can apply (2.5) to obtain

$$\Pr(Y_{\mathcal{G}} = 0) = O(\Pr(Y = 0)) = o(1),$$

completing the proof. \square

2.3 Structure of the rest of the paper

We assume that $s \geq 3$, since the results for $s = 2$ were proved by Frieze et al. [8] and Janson [11]. It remains to investigate the second moment of Y and the interaction of Y with short cycles.

Section 3 contains some terminology and preliminary results, and describes a common framework which we will use for the calculations in the following two sections.

In Section 4 we calculate $\mathbb{E}(YX_k)/\mathbb{E}Y$, using a generating function to assist in our calculations. For each $k \geq 1$ this determines the value of δ_k such that $\mathbb{E}(YX_k)/\mathbb{E}Y$ tends to $\lambda_k(1 + \delta_k)$, where λ_k is defined in (2.8). Standard arguments imply that Theorem 2.3 (A2) also holds, which allows us to prove Theorem 1.2.

The remainder of the paper is devoted to completing the small subgraph conditioning argument to prove Theorem 1.1. In Section 4.1 we calculate $\sum_{k=1}^{\infty} \lambda_k \delta_k^2$, proving that assumption (A1) of Theorem 2.3 holds. Section 5 contains the analysis of the second moment $\mathbb{E}(Y^2)$. Here we use Laplace summation to find an asymptotic expression for the second moment, proving that (A4) of Theorem 2.3 holds. This involves proving that a certain 4-variable real function has a unique maximum in a certain bounded convex domain. (This optimisation is performed in Section A.) Finally, the proof of Theorem 1.1 is completed in Section 6.

3 Terminology and common framework

For the small subgraph conditioning method, we need to calculate the second moment of Y and establish condition (A2) of Theorem 2.3. We now describe a common framework which we will use for these calculations, which will be completed in Sections 4 and 5.

Suppose that F_C, F_H are both subpartitions of some partition in $\Omega(n, r, s)$, such that $G(F_H)$ is a loose Hamilton cycle and $G(F_C)$ is a loose k -cycle. In particular, $|F_H| = t$ and $|F_C| = k$. Write H for $G(F_H)$ to C for $G(F_C)$. We will be particularly interested in two extreme cases, namely, when k is constant or when $k = t$. (In the latter case, C is also a loose Hamilton cycle.) In order to describe the common framework we will use for our calculations in these cases, we need some terminology. We will use Figure 2 as a running example: it shows a 12-cycle C in a 5-uniform hypergraph, and some edges of a Hamilton cycle H .

Suppose that there are $k - a$ parts in $F_C \cap F_H$, and that $G(F_C \cap F_H)$ forms b connected components in $G(H)$, each of which is a path. Then there are a parts in $F_C \setminus F_H$. In our running example from Figure 2, the edges shown in bold belong to $G(F_C \cap F_H)$. There are 5 of them, in 3 paths, so $a = 12 - 5 = 7$ and $b = 3$. The 7 edges of $G(F_C \setminus F_H)$ are shown as thin rectangles. The dashed lines indicate partial edges which belong to $G(F_H \setminus F_C)$.

Let v be a vertex in a loose cycle C . If v has degree 2 in C then we will say that v is *C-external* (or just *external*, if no confusion can arise). Otherwise, v has degree 1

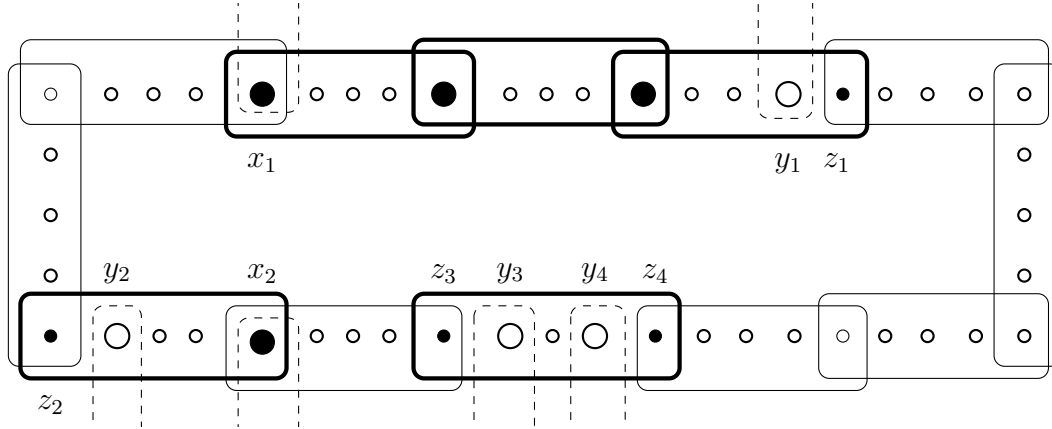


Figure 2: A 12-cycle C with 5 edges in $G(F_C \cap F_H)$, in three paths.

in C and we will say that it is C -internal (or just *internal*). A loose Hamilton cycle H has t external vertices and $(s - 2)t$ internal vertices, where, recall, $t = n/(s - 1)$ is the number of edges in H . In Figure 2, vertices x_1 and x_2 (shown as large black circles) are C -external and H -external. Vertices z_1, z_2, z_3 and z_4 (shown as small black circles) are C -external and H -internal. Finally, vertices y_1, y_2, y_3 and y_4 (shown as large white circles) are H -external and C -internal. It will be important to know whether a given vertex is external or internal in C and/or H .

The edges of $G(F_C \cap F_H)$ which start or end a component of $G(F_C \cap F_H)$ will play a special role: we call these *terminal* edges. If a component of $G(F_C \cap F_H)$ has length at least two then it has two terminal edges, and each terminal edge contains precisely one C -external vertex which is incident with an edge of $G(F_C \setminus F_H)$. Such a vertex is called a *connection vertex*. In Figure 2 there is one component of $G(F_C \cap F_H)$ which has more than one edge (and hence has two distinct terminal edges). The connection vertices for this component are x_1 and z_1 .

On the other hand, if a component of $G(F_C \cap F_H)$ has length 1 then it has only one terminal edge, containing two connection vertices. In Figure 2 there are two such components: one has connection vertices z_2 and x_2 , and the other has connection vertices z_3 and z_4 . We refer to components of $G(F_C \cap F_H)$ of length one as *1-components*.

As we will see later, the two connection vertices in components of $G(F_C \cap F_H)$ of length at least two are essentially independent, as far as our counting argument is concerned, since they belong to *distinct* terminal edges. This is not true for the 1-components in $G(F_C \cap F_H)$, so we need to take special care with these. (This is the main reason why the hypergraph calculations are substantially more difficult than the graph case.)

3.1 Templates

Starting with a set of parts F_H corresponding to an arbitrary Hamilton cycle H , we need to consider all possible choices for a set F_C of parts corresponding to a k -cycle C and then, all possible completions of $F_H \cup F_C$ to a partition in $\mathcal{F}(n, r, s)$. We will do this with the help of templates, which we now define.

Suppose that F_C is given. If $F_C \cap F_H$ is neither empty nor equal to F_H , then we fix a start-vertex v which is a C -external vertex that belongs to precisely one edge e of $G(F_C \cap F_H)$. This uniquely determines a direction around C in which e is the first edge of C (starting from v). Otherwise, let v be any C -external vertex and choose an arbitrary direction around C . Then the *template* θ for F_C , with respect to the chosen start-vertex and direction, is the sequence $\theta = (\theta_1, \dots, \theta_k) \in \{0, 1\}^k$ such that the j 'th coordinate of θ is 1 if and only if the j 'th part of F_C belongs to F_H . Note that, by definition, either $\theta_1 = 1$ and $\theta_k = 0$, or $\theta_1 = \theta_2 = \dots = \theta_k$.

Let b be the number of indices $j \in \{1, \dots, k\}$ such that $\theta_j = 1$ and $\theta_{j+1} = 0$. If $b \geq 1$ then the template determines a sequence $\ell = (\ell_1, \dots, \ell_b)$ of lengths of the components of $G(F_C \cap F_H)$, in order around C (from the given start-vertex, in the given direction). Here the start-vertex of C is a connection vertex and the first edge of C is a terminal edge of one of the components of $G(F_C \cap F_H)$. The template also determines the sequence $\mathbf{u} = (u_1, \dots, u_b)$ of “gap lengths”, namely, the number of edges between consecutive components of $G(F_C \cap F_H)$, around C . Here u_j is the number of edges of $G(F_C \setminus F_H)$ between the j th and $(j+1)$ th components of $G(F_C \cap F_H)$ (and u_b is the number of edges between the last and the first component). In Figure 2, if we take x_2 to be the start-vertex and choose the clockwise direction, then the template for F_C is $(1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0)$. The sequence of intersection lengths is $\ell = (1, 3, 1)$ and the sequence of gap lengths is $\mathbf{u} = (2, 4, 1)$.

Conversely, suppose that $b \geq 1$ and let ℓ, \mathbf{u} be two sequences of b positive integers which together sum to k . Then (ℓ, \mathbf{u}) determines a template θ for F_C , with respect to the chosen start vertex and direction. (The first ℓ_1 entries of θ are 1, the next u_1 symbols are zero, the next ℓ_2 entries are 1, and so on.)

When $b = 0$, either $F_C = F_H$ with template $\theta = (1, 1, \dots, 1)$, or $F_C \cap F_H = \emptyset$ with template $\theta = (0, 0, \dots, 0)$. The case $b = 0$ will be considered separately in many of our arguments.

We let c denote the number of intersection lengths which are at least two, so $b - c$ counts the number of intersection lengths equal to 1 (that is, the number of entries of ℓ which equal 1).

3.2 A common framework

Recall that X_k denotes the number of loose k -cycles in $\mathcal{F}(n, r, s)$, for $k \geq 2$, and X_1 is the number of loops (parts containing more than one point from some cell). We now describe the common framework that we will use when calculating $\mathbb{E}(YX_k)$, with

$k = O(1)$, and $\mathbb{E}(Y^2) = \mathbb{E}(YX_t)$. These calculations are presented in Sections 4 and 5, respectively. Let $F \in \mathcal{F}(n, r, s)$ and write

$$\mathbb{E}(YX_k) = \sum_{(F_H, F_C)} \Pr(F_C \cup F_H \subseteq F)$$

where $G(F_H)$ is a loose Hamilton cycle and $G(F_C)$ is a loose k -cycle. We will perform the summation using the following steps.

Step 1: Choose F_H .

We must choose a loose Hamilton cycle H , and count the number of ways to choose parts F_H to correspond to the edges of H . Recalling (2.1), the number of choices of F_H is given by

$$\frac{n!}{2t((s-2)!)^t} r^n (r-1)^t = \frac{p(rn)}{p(rn-st)} \mathbb{E}Y,$$

using (2.6) and the arguments given in the proof of Lemma 2.1.

Step 2: Choose a template for F_C .

When $b \geq 1$, we choose a vector $\ell = (\ell_1, \dots, \ell_b)$ and a vector $\mathbf{u} = (u_1, \dots, u_b)$ of intersection lengths and gap lengths, respectively. This uniquely determines the corresponding template $\boldsymbol{\theta}$, and also determines the parameters a and c ; that is, the number of parts in $F_C \setminus F_H$, and the number of intersection paths of length at least two, respectively. Denote the overall number of choices of templates for given values of (a, b, c) by $M_2(k, a, b, c)$. When $k = O(1)$ we calculate $M_2(k, a, b, c)$ in Section 4, while the case that $k = t$ is analysed in Section 5.

When $b = 0$ there are two valid cases. We have $M_2(k, k, 0, 0) = 1$, since this corresponds to the case that $F_C \cap F_H = \emptyset$, which has the unique template $\boldsymbol{\theta} = (0, 0, \dots, 0)$. Similarly,

$$M_2(k, 0, 0, 0) = \begin{cases} 1 & \text{if } k = t, \\ 0 & \text{otherwise,} \end{cases}$$

since this case can only arise when $k = t$ and $F_H = F_C$, corresponding to the unique template $\boldsymbol{\theta} = (0, 0, \dots, 0)$.

Step 3: Identify $F_H \cap F_C$ and order components.

First suppose that $b \geq 1$. We choose a permutation σ of $[b]$ and select a sequence of b vertex-disjoint induced subhypergraphs (paths) from H with lengths $\ell_{\sigma(1)}, \dots, \ell_{\sigma(b)}$ in order around H . The j 'th component around H becomes the $\sigma(j)$ 'th component around C (with respect to the fixed start vertex and direction

on C), matching the template for F_C . This determines the parts of $F_C \cap F_H$, together with an ordering of the components of $G(F_C \cap F_H)$ around C , as a sequence. In Lemma 3.2 below, we prove that there are

$$\frac{t(t-k+a-1)!}{(t-k+a-b)!} \quad (3.1)$$

ways to do this. (The orientation of these components is performed in Step 4.) If $b = 0$ then there is nothing to do in this step.

Step 4: Choose the connection vertices and corresponding points.

Now we determine the identity of the connection vertices, choose a point corresponding to each of these vertices and determine the orientation of each component of $G(F_C \cap F_H)$ within C . We prove in Lemma 3.3 below that there are

$$(2h(r, s))^b \left(\frac{(rs - r - s)^2}{h(r, s)} \right)^c \quad (3.2)$$

ways to do this, where

$$h(r, s) = (r-2)^2 + 2(s-2)(r-1)(r-2) + \frac{1}{2}(s-2)(s-3)(r-1)^2. \quad (3.3)$$

Observe that $h(r, s) > 0$ whenever $s \geq 3$ and $r \geq 4$.

Again, if $b = 0$ then there is nothing to do in this step, and (3.2) gives the factor 1.

Step 5: Choose the rest of $F_C \setminus F_H$ and adjust for overcounting.

Using the template θ , we must identify all vertices in edges of $G(F_C \setminus F_H)$ other than the connection vertices, and assign points to these vertices, thereby completing all parts in $F_C \setminus F_H$. Finally, we just adjust our counting so that each choice of F_C only arises once for a given choice of F_H : we achieve this by dividing by the number of templates corresponding to F_C . Let $M_5(k, a, b)$ denote the number of ways to perform this step: it will turn out that this number is independent of c . We calculate $M_5(k, a, b)$ in Section 4 when $k = O(1)$, and in Section 5 when $k = t$.

Step 6: Multiply by the probability of containing $F_C \cup F_H$.

By now, all parts in $F_C \cup F_H$ have been specified. So we multiply by the probability that the specified parts are contained in $F \in \mathcal{F}(n, r, s)$, namely,

$$\frac{p(rn - s(t+a))}{p(rn)} = \frac{p(rn - s(t+a))}{p(rn - st)} \cdot \frac{p(rn - st)}{p(rn)}.$$

We express the probability in this form to display the factor which arises in Step 1.

The case $k = 1$ is somewhat special, since X_1 counts all 1-cycles, not just loose 1-cycles. This will be discussed further in Section 4.

Combining Steps 1 to 6 leads to the following expression:

$$\begin{aligned} \frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} &= M_5(k, 0, 0) + M_5(k, k, 0) \cdot \frac{p(rn - s(t + k))}{p(rn - st)} \\ &\quad + \sum_{\substack{a \geq b \geq 1, \\ c \geq 0}} M_2(k, a, b, c) \cdot \frac{t(t - k + a - 1)!}{(t - k + a - b)!} \cdot (2h(r, s))^b \cdot \left(\frac{(rs - r - s)^2}{h(r, s)} \right)^c \\ &\quad \times M_5(k, a, b) \cdot \frac{p(rn - s(t + a))}{p(rn - st)}. \end{aligned} \quad (3.4)$$

Before we prove (3.1) and (3.2), we state the following lemma which contains two useful combinatorial facts (proof omitted). We adopt the convention that $\binom{0}{0} = 1$.

Lemma 3.1. *Let R, T be positive integers with $R \leq T$, and let J be a nonnegative integer.*

(i) *The number of sequences of R positive integers which sum to T is*

$$\binom{T-1}{R-1}.$$

(ii) *The number of sequences of R positive integers which sum to T and which contain precisely J entries equal to 1 is*

$$\binom{R}{J} \binom{T-R-1}{R-J-1}$$

if $R < T$, and equals 1 if $R = T$ (in which case also $J = T$).

In the next two lemmas, H is a fixed Hamilton cycle. First we calculate the number of ways to perform Step 3 when $b \geq 1$.

Lemma 3.2. *Suppose that a template for F_C has been fixed, with $b \geq 1$. The number of ways to fix a permutation σ of $[b]$ and choose b vertex-disjoint induced subhypergraphs (paths) of H with lengths $\ell_{\sigma(1)}, \dots, \ell_{\sigma(b)}$ in order around H is given by (3.1).*

Proof. Fix an H -external start-vertex w and direction on H , in $2t$ ways. We will ensure that w belongs to precisely one edge e of $G(F_C \cap F_H)$, namely, the first edge around H from w in the chosen direction. Choose a permutation σ of $[b]$, in $b!$ ways. Let ℓ denote the sequence of intersection lengths corresponding to the fixed template. To choose paths of length $\ell_{\sigma(1)}, \dots, \ell_{\sigma(b)}$ in this order around H , it suffices to choose a sequence of (positive) integers (g_1, \dots, g_b) , which will be the gap lengths around H in order. That

is, the first $\ell_{\sigma(1)}$ parts of F_H (in the chosen direction, starting from w) will form a component of $G(F_C \cap F_H)$, and then the next g_1 parts of F_H will belong to $F_H \setminus F_C$, and so on.

Let $k - a$ be the number of edges in $G(F_C \cap F_H)$, as determined by the template. Then the gap lengths around H must add up to $t - k + a$. By Lemma 3.1(i), there are $\binom{t-k+a-1}{b-1}$ ways to select these gap lengths around H , which determine the b paths around H with the given lengths. Finally we divide by $2b$, the number of choices of start-vertex and direction on H which lead to the same choice of edges in $G(F_C \cap F_H)$. Multiplying these factors together gives

$$\frac{t(t-k+a-1)!}{(t-k+a-b)!},$$

completing the proof. \square

Next we calculate the number of ways to perform Step 4 when $b \geq 1$.

Lemma 3.3. *Suppose that a template for F_C has been fixed, with $b \geq 1$. The number of ways to select the $2b$ connection vertices, to assign a point to each, and to orient each component of $G(F_C \cap F_H)$ within C is given by (3.2).*

Proof. Each connection vertex is incident with one edge of $G(F_C \cap F_H)$, which we denote by e , and one edge of $G(F_C \setminus F_H)$, which we denote by \hat{e} . We must choose the connection vertex and assign a point to it in the part corresponding to the edge \hat{e} . Firstly, suppose that v is a connection vertex in a terminal edge e which belongs to a component of $G(F_C \cap F_H)$ of length at least 2. If v equals the H -external vertex in e then there is 1 choice for the vertex, and $r - 2$ ways to select a point corresponding to this vertex (in the part corresponding to \hat{e}). Otherwise, there are $s - 2$ H -internal vertices which can be chosen for v , and $r - 1$ ways to assign a point corresponding to this vertex. Overall, this gives $r - 2 + (s - 2)(r - 1) = rs - r - s$ ways to choose the connection vertex v and a point corresponding to v in the edge \hat{e} . The choice of v has no effect on the number of choices for the connection vertex in the other terminal edge of this component, so we can simply square this contribution to take both connection vertices into account, giving a contribution of $(rs - r - s)^2$ in this case.

It remains to consider connection vertices which belong to 1-components of $G(F_C \cap F_H)$. The two connection vertices in e may both be C -external, giving 1 choice (for the unordered pair of connection vertices) and $(r - 2)^2$ ways to assign the corresponding points. There are $2(s - 2)(r - 1)(r - 2)$ choices if one connection vertex in e is H -external and the other is H -internal. (For example, see the edge containing vertices z_2 and x_2 in Figure 2.) Finally, if both connection vertices in e are H -internal then there are $\frac{1}{2}(s - 2)(s - 3)(r - 1)^2$ choices for the connection vertices (as an unordered pair) and the corresponding points. (See the edge containing vertices z_3 and z_4 in Figure 2.) So the contribution in the second case is $h(r, s)$, as defined in (3.3).

Now that all connection vertices and their points have been identified, we decide the orientation of each component of $G(F_C \cap F_H)$ within C by ordering the two connection vertices within each component. That is, we decide which connection vertex within a given component should be the “first” one we meet as we move around C in the specified direction. There are 2^b choices of orientation.

Overall, the number of ways to select the $2b$ connection vertices, to assign a point to each, and to orient each component of $G(F_C \cap F_H)$ within C is

$$2^b (rs - r - s)^{2c} h(r, s)^{b-c} = (2h(r, s))^b \left(\frac{(rs - r - s)^2}{h(r, s)} \right)^c,$$

as required. \square

To apply (3.4), it remains to calculate $M_2(k, a, b, c)$ and $M_5(k, a, b)$ and perform the summation, in the two extreme cases, namely when $k = O(1)$ (in Section 4) and $k = t$ (in Section 5). Several simplifications make the calculations easier when k is constant, allowing the use of generating functions to assist us with Steps 2 and 4. When $k = t$ we use Laplace summation to calculate the sum over all templates. This will involve detailed analysis of a certain real function of four variables.

4 Effect of short cycles

We use a generating function to perform Step 2 for short cycles.

Lemma 4.1. *Suppose that $k \geq 1$ is fixed. If $a, b \geq 1$ and $c \geq 0$ then number of templates for F_C (Step 2) with parameters (a, b, c) is*

$$M_2(k, a, b, c) = [x^k y^a z^b w^c] \left(\frac{x^2 y z (1 - x + xw)}{(1 - x)(1 - xy)} \right)^b.$$

Proof. We construct a generating function with the following variables:

- x marks the number of edges of C ,
- y marks the number of parts in $F_C \setminus F_H$,
- z marks the number of components in $G(F_C \cap F_H)$,
- w marks the number of components in $G(F_C \cap F_H)$ of length at least 2.

Since $b \geq 1$, the template θ starts with some positive number of entries equal to 1, marking off the length of the first component of $G(F_C \cap F_H)$. If the first component has length 1 then it contributes one edge to C and it contributes one component, so it is stored as xz . Otherwise, the first component gives one component, which is of length

at least 2 and consists of $j + 2$ edges, for some $j \geq 0$. We store this in the generating function as $x^2zw/(1-x)$. Therefore the contribution of the first component of $F_C \cap F_H$ to the generating function is

$$xz + \frac{x^2zw}{1-x} = \frac{xz(1-x+xw)}{1-x}.$$

Next we must specify the first gap length, which must be at least 1. If the gap length is j then this gap contributes j edges to C and j edges to $G(F_C \setminus F_H)$, so we store this in the generating function as

$$\frac{xy}{1-xy}.$$

To complete the template we repeat the above procedure b times in total. Hence when $b \geq 1$ we have

$$M_2(k, a, b, c) = [x^k y^a z^b w^c] \left(\frac{x^2yz(1-x+xw)}{(1-x)(1-xy)} \right)^b.$$

□

Next we perform Step 5. Recall that during Steps 1–4 we have identified F_H , the template θ , the parts in $F_C \cap F_H$ and the connection vertices. In Step 5 we determine the rest of $F_C \setminus F_H$ and adjust for overcounting, usually by dividing by the number of templates corresponding to a given F_C .

Lemma 4.2. *Let $k \geq 1$ be a fixed integer and let a, b, c be nonnegative integers with $c \leq b \leq k - a$, such that if $b = 0$ then $a = k$. If $b \geq 1$ then $M_5(k, a, b)$ is asymptotically equal to*

$$\frac{1}{2b} \left(\frac{(r-2)(rs-r-s-1)(rs-r-s)^{s-2}t^{s-1}}{(s-2)!} \right)^a ((r-2)(rs-r-s-1)t)^{-b}$$

while $M_5(k, k, 0)$ is asymptotically equal to

$$\frac{1}{2k} \left(\frac{(r-2)(rs-r-s-1)(rs-r-s)^{s-2}t^{s-1}}{(s-2)!} \right)^k.$$

Finally, $M_5(k, 0, 0) = 0$.

Proof. The last statement is clear since the case $a = 0$ can only arise when $k = t$ and $F_H = F_C$, so this case is ruled out by the assumption that $k = O(1)$.

Next, consider the case that $k = 1$. The random variable X_1 counts all 1-cycles, not just loose 1-cycles. But loose 1-cycles involve $s-1$ distinct cells, while non-loose 1-cycles involve at most $s-2$ distinct cells. This implies that the contribution to $M_5(1, 1, 0)$

from non-loose 1-cycles is $O(1/n)$ times the contribution to $M_5(1, 1, 0)$ from loose 1-cycles only. Hence when $k = 1$, it suffices to only consider loose 1-cycles ($a = k = 1$ and $b = 0$). As X_k counts the number of loose k -cycles, when $k \geq 2$, this brings the $k = 1$ case in line with the general case, so we may treat them both together below.

Let θ be the template which was chosen for F_C in Step 2. We must identify all C -external vertices which are not incident with an edge of $G(F_C \cap F_H)$ (there are $a - b$ of them), and all C -internal vertices which are not incident with an edge of $G(F_C \cap F_H)$ (there are $(s - 2)a$ of them). We call the vertices identified in this step the *new* vertices. This will specify the identity of all remaining vertices in $G(F_C \setminus F_H)$. The template θ determines a fixed start-vertex and direction around C . Since k is constant, as we move around the cycle C identifying vertices, there are always $t - O(1) \sim t$ remaining H -external vertices to choose from, and there are always $n - t - O(1) \sim (s - 2)t$ remaining H -internal vertices to choose from. For a vertex v which we have just identified, the number of choices for points representing v in parts corresponding to the edges of $G(F_C \setminus F_H)$ incident with v is

$$\begin{cases} (r - 2)(r - 3) & \text{if } v \text{ is } C\text{-external and } H\text{-external,} \\ (r - 1)(r - 2) & \text{if } v \text{ is } C\text{-external and } H\text{-internal,} \\ (r - 2) & \text{if } v \text{ is } C\text{-internal and } H\text{-external,} \\ (r - 1) & \text{if } v \text{ is } C\text{-internal and } H\text{-internal.} \end{cases}$$

This gives a *sequence* of C -internal vertices for each edge, but we need a *set*. If an edge in $G(F_C \setminus F_H)$ has j C -internal vertices which are H -external, then we must divide by $j!(s - 2 - j)! = (s - 2)! / \binom{s - 2}{j}$.

Hence the number of ways to identify the new C -external vertices of $G(F_C \setminus F_H)$, and assign points to them, is asymptotically equal to

$$\begin{aligned} \sum_{d=0}^{a-b} \binom{a-b}{d} ((r - 2)(r - 3)t)^d ((r - 1)(r - 2)(s - 2)t)^{a-b-d} \\ = ((r - 2)(rs - r - s - 1)t)^{a-b} \end{aligned} \quad (4.1)$$

and the number of ways to identify the C -internal vertices of $G(F_C \setminus F_H)$, and assign points to them, is asymptotically equal to

$$\begin{aligned} \left(\frac{1}{(s - 2)!} \sum_{j=0}^{s-2} \binom{s-2}{j} ((r - 2)t)^j ((r - 1)(s - 2)t)^{s-2-j} \right)^a \\ = \left(\frac{(rs - r - s)^{s-2} t^{s-2}}{(s - 2)!} \right)^a. \end{aligned} \quad (4.2)$$

If $b \geq 1$ then each choice of F_C corresponds to precisely $2b$ templates. So the expression for $M_5(k, a, b)$ follows by multiplying (4.1) and (4.2) and dividing by $2b$.

If $b = 0$ then $a = k$ and the argument above determines a sequence of parts of F_C , with respect to some given start-vertex and direction. We must divide by $2k$ to adjust for this multiple counting. Indeed, the choice of direction really corresponds to the choice of one of the two points representing the start vertex v in C . So we must also divide by 2 when $k = 1$, to replace the ordered pair of points representing v by an unordered pair of points. This completes the proof. \square

We now have all the information we need in order to perform the summation in (3.4).

Lemma 4.3. *Let $s \geq 2$ and $r \geq 2$, with $(r, s) \neq (2, 2)$. For any fixed integer $k \geq 1$ we have*

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \sim \frac{((r-1)(s-1))^k}{2k} + \frac{\zeta_1^k}{2k} + \frac{\zeta_2^k}{2k} - \frac{1}{2k}$$

where $\zeta_1, \zeta_2 \in \mathbb{C}$ satisfy

$$\zeta_1 + \zeta_2 = -\frac{rs^2 - s^2 - 2rs + r + 2}{rs - r - s}, \quad \zeta_1 \zeta_2 = \frac{(s-1)(s-2)(r-1)}{rs - r - s}. \quad (4.3)$$

Proof. Fix $k \geq 1$. Before applying Lemmas 4.1 and 4.2, we simplify some factors of (3.4). Since $k - a = O(1)$, the factor of (3.4) from Step 3 equals

$$\frac{t(t-k+a-1)!}{(t-k+a-b)!} \sim t^b.$$

Similarly, using Stirling's formula, the factor of (3.4) from Step 6 equals

$$\frac{p(rn - st - sa)}{p(rn - st)} \sim \left(\frac{(s-1)!}{(rs - r - s)^{s-1} t^{s-1}} \right)^a.$$

Combining these with Lemmas 4.1 and 4.2, the expression (3.4) becomes

$$\begin{aligned} \frac{\mathbb{E}(YX_k)}{\mathbb{E}(Y)} &\sim \frac{\mu_1^k}{2k} + \sum_{\substack{a \geq b \geq 1, \\ c \geq 0}} [x^k y^a z^b w^c] \frac{1}{2b} \left(\frac{x^2 y z (1-x+xw)}{(1-x)(1-xy)} \right)^b \mu_1^a \mu_2^b \mu_3^c \\ &= \frac{\mu_1^k}{2k} - \frac{1}{2} \sum_{a, c \geq 0} [x^k y^a w^c] \ln \left(1 - \frac{\mu_1 \mu_2 x^2 y (1-x + \mu_3 x w)}{(1-x)(1-\mu_1 xy)} \right), \end{aligned}$$

where

$$\left. \begin{aligned} \mu_1 &= \frac{(s-1)(r-2)(rs-r-s-1)}{rs-r-s} \\ \mu_2 &= \frac{2h(r,s)}{(r-2)(rs-r-s-1)} \\ \mu_3 &= \frac{(rs-r-s)^2}{h(r,s)}. \end{aligned} \right\} \quad (4.4)$$

The summation over a and c can be achieved by setting $y = w = 1$, giving

$$\begin{aligned} \frac{\mathbb{E}(Y X_k)}{\mathbb{E}(Y)} &\sim \frac{\mu_1^k}{2k} - \frac{1}{2} [x^k] \ln \left(1 - \frac{\mu_1 \mu_2 x^2 (1 - (1 - \mu_3)x)}{(1-x)(1 - \mu_1 x)} \right) \\ &= -\frac{1}{2} [x^k] \ln \left(\frac{1 - (\mu_1 + 1)x - \mu_1(\mu_2 - 1)x^2 - \mu_1 \mu_2 (\mu_3 - 1)x^3}{1-x} \right) \\ &= -\frac{1}{2} [x^k] \ln \left(\frac{(1 - (r-1)(s-1)x) \left(1 + \frac{rs^2 - s^2 - 2rs + r + 2}{rs - r - s}x + \frac{(s-1)(s-2)(r-1)}{rs - r - s}x^2 \right)}{1-x} \right) \end{aligned}$$

using (4.4) for the final equality. The quadratic factor inside the logarithm factors as

$$1 + \frac{rs^2 - s^2 - 2rs + r + 2}{rs - r - s}x + \frac{(s-1)(s-2)(r-1)}{rs - r - s}x^2 = (1 - \zeta_1 x)(1 - \zeta_2 x)$$

where the roots $\zeta_1, \zeta_2 \in \mathbb{C}$ are defined by (4.3). Using this factorisation we can write

$$\begin{aligned} \frac{\mathbb{E}(Y X_k)}{\mathbb{E}(Y)} &\sim -\frac{1}{2} [x^k] \left(\ln(1 - (r-1)(s-1)x) + \ln(1 - \zeta_1 x) + \ln(1 - \zeta_2 x) - \ln(1-x) \right) \\ &= \frac{((r-1)(s-1))^k}{2k} + \frac{\zeta_1^k}{2k} + \frac{\zeta_2^k}{2k} - \frac{1}{2k}, \end{aligned}$$

as claimed. \square

The following corollary follows immediately from Lemma 2.4, (2.9) and Lemma 4.3.

Corollary 4.4. *Suppose that $r, s \geq 2$ are fixed integers with $(r, s) \neq (2, 2)$. Then condition (A2) of Theorem 2.3 holds with λ_k given by (2.8) and δ_k defined by*

$$\delta_k = \frac{\zeta_1^k + \zeta_2^k - 1}{((r-1)(s-1))^k} \quad (4.5)$$

for $k \geq 1$.

Observe that even though ζ_1, ζ_2 may be complex, δ_k is always real, by de Moivre's Theorem.

Now we can prove Theorem 1.2, giving the expected number of loose Hamilton cycles in a random r -regular s -uniform hypergraph when $r, s \geq 2$ are fixed integers and $(r, s) \neq (2, 2)$.

Proof of Theorem 1.2. The result for $s = 2$ was proved by Janson [11, Theorem 2]. Now assume that $s \geq 3$. The hypergraph $G(F)$ is simple if and only if F has no 1-cycles (that is, $X_1 = 0$) and no repeated edges. The probability that two parts of $F \in \mathcal{F}(n, r, s)$ give rise to a repeated edge is

$$O\left(\frac{n^s p(rn - 2s)}{p(rn)}\right) = O(n^{2-s}) = o(1) \quad (4.6)$$

when $s \geq 3$. In other words, a.a.s. $G(F)$ has no repeated edges and it suffices to condition on $X_1 = 0$. Corollary 4.4 states that condition (A2) of Theorem 2.3 holds, which implies that

$$\frac{\mathbb{E}Y_G}{\mathbb{E}Y} \sim \frac{\mathbb{E}(Y \mid X_1 = 0)}{\mathbb{E}Y} \longrightarrow e^{-\lambda_1 \delta_1}.$$

We calculate

$$-\lambda_1 \delta_1 = \frac{1}{2} (1 - \zeta_1 - \zeta_2) = \frac{(s-1)(rs-s-2)}{2(rs-r-s)},$$

using (4.3). Combining this with Corollary 2.2 completes the proof. \square

4.1 Preparation for small subgraph conditioning

Before proceeding to the second moment calculations, we establish some results which we will be needed in order to apply Theorem 2.3. Recall the definition of ζ_1, ζ_2 from (4.3), and the definition of δ_k from (4.5).

The first result shows that $\delta_k > -1$ for all $k \geq 1$. As we will see, this will be needed in the proof of the threshold result, Theorem 1.1.

Lemma 4.5. *Suppose that $r, s \geq 2$ with $(r, s) \neq (2, 2)$. Then $\delta_k > -1$ for all $k \geq 1$.*

Proof. For ease of notation, write $A = (r-1)(s-1)$. The desired inequality is equivalent to

$$A^k + \zeta_1^k + \zeta_2^k - 1 > 0. \tag{4.7}$$

First suppose that ζ_1 and ζ_2 are not real. In this case, ζ_1 and ζ_2 form a complex conjugate pair, and

$$\begin{aligned} A^k + \zeta_1^k + \zeta_2^k - 1 &\geq A^k - |\zeta_1 \zeta_2|^{k/2} - 1 \\ &= A^k - 2 \left(\frac{(r-1)(s-1)(s-2)}{rs-r-s} \right)^{k/2} - 1 \end{aligned}$$

using (4.3). This expression is positive for all $r, s \geq 2$ and $k \geq 1$.

For the remainder of the proof, suppose that ζ_1 and ζ_2 are real. Then (4.7) holds for all even $k \geq 2$. We will prove by induction that

$$|\zeta_1|^k + |\zeta_2|^k < A^k - 1 \tag{4.8}$$

for all $k \geq 1$. Since ζ_1 and ζ_2 are both negative, by (4.3), we see that (4.7) and (4.8) are equivalent when k is odd.

Firstly, note that (4.8) holds when $k = 1$, by (4.3). Now suppose that $k \geq 2$, and observe that

$$\zeta_1^k + \zeta_2^k = (\zeta_1 + \zeta_2) (\zeta_1^{k-1} + \zeta_2^{k-1}) - \zeta_1 \zeta_2 (\zeta_1^{k-2} + \zeta_2^{k-2}).$$

The terms $\zeta_1^k + \zeta_2^k$, $(\zeta_1 + \zeta_2)(\zeta_1^{k-1} + \zeta_2^{k-1})$ and $\zeta_1\zeta_2(\zeta_1^{k-2} + \zeta_2^{k-2})$ have the same sign, since ζ_1 and ζ_2 are both negative. Therefore, by the inductive hypothesis,

$$\begin{aligned} |\zeta_1|^k + |\zeta_2|^k &= |\zeta_1^k + \zeta_2^k| \leq |(\zeta_1 + \zeta_2)(\zeta_1^{k-1} + \zeta_2^{k-1})| \\ &= (|\zeta_1| + |\zeta_2|)(|\zeta_1|^{k-1} + |\zeta_2|^{k-1}) \\ &< (A - 1)(A^{k-1} - 1) \\ &\leq A^k - 1, \end{aligned}$$

since $A^{k-1} + A \geq 2$ when $r, s \geq 2$. This completes the inductive step. \square

Next, we show that condition (A3) of Theorem 2.3 holds, under weak conditions on r and s .

Lemma 4.6. *Let $s \geq 3$ and $r \geq s + 1$. With λ_k as defined in (2.8) and δ_k as defined in (4.5), we have*

$$\exp\left(\sum_{k \geq 1} \lambda_k \delta_k^2\right) = \frac{r(rs - r - s)}{(r - 2)\sqrt{Q(r, s)}} \quad (4.9)$$

where

$$Q(r, s) = r^2s^2 - rs^3 - 2r^2s + 3rs^2 + s^3 + r^2 - 6rs + 4r - 4s + 4.$$

Furthermore, $Q(r, s) > 0$ so condition (A3) of Theorem 2.3 holds.

Proof. Again we write $A = (r - 1)(s - 1)$ for ease of notation. We calculate

$$\begin{aligned} \sum_{k \geq 1} \lambda_k \delta_k^2 &= \sum_{k \geq 1} \frac{1}{2k A^k} (\zeta_1^k + \zeta_2^k - 1)^2 \\ &= \sum_{k \geq 1} \frac{1}{2k A^k} (\zeta_1^{2k} + \zeta_2^{2k} + 2(\zeta_1\zeta_2)^k - 2\zeta_1^k - 2\zeta_2^k + 1) \\ &= -\frac{1}{2} \ln(1 - \zeta_1^2/A) - \frac{1}{2} \ln(1 - \zeta_2^2/A) - \ln(1 - \zeta_1\zeta_2/A) \\ &\quad + \ln(1 - \zeta_1/A) + \ln(1 - \zeta_2/A) - \frac{1}{2} \ln(1 - 1/A). \end{aligned}$$

Therefore

$$\begin{aligned} \exp\left(\sum_{k \geq 1} \lambda_k \delta_k^2\right) &= \left(\frac{A(A - \zeta_1)^2(A - \zeta_2)^2}{(A - \zeta_1^2)(A - \zeta_2^2)(A - 1)(A - \zeta_1\zeta_2)^2}\right)^{1/2} \\ &= \left(\frac{A(A^4 - 2(\zeta_1 + \zeta_2)A^3 + ((\zeta_1 + \zeta_2)^2 + 2\zeta_1\zeta_2)A^2 - 2\zeta_1\zeta_2(\zeta_1 + \zeta_2)A + (\zeta_1\zeta_2)^2)}{(A - 1)(A - \zeta_1\zeta_2)^2(A^2 - ((\zeta_1 + \zeta_2)^2 - 2\zeta_1\zeta_2)A + (\zeta_1\zeta_2)^2)}\right)^{1/2}. \end{aligned}$$

Substituting for A and for $\zeta_1 + \zeta_2$ and $\zeta_1\zeta_2$ leads to (4.9) after much simplification, using (4.3).

For the final statement, rewrite Q as

$$Q(r, s) = (s-1)^2 r^2 - (s-1)(s^2 - 2s + 4)r + s^3 - 4s + 4.$$

For fixed $s \geq 2$, the roots of this quadratic in r occur at

$$\frac{s^2 - 2s + 4 \pm s\sqrt{(s-2)(s-6)}}{2(s-1)}.$$

It follows that $Q(r, s)$ is positive whenever $r \geq \frac{s^2 - s + 2}{s-1}$, and since $s \geq 3$ this inequality holds whenever $r \geq s + 1$. \square

5 The second moment

In this section we calculate the second moment of Y . We use the framework from Section 3, but write F_1 and F_2 rather than F_H and F_C , respectively, and let $H_j = G(F_j)$ for $j = 1, 2$.

First we state the following combinatorial fact without proof.

Lemma 5.1. *Let J, R, T be positive integers and let*

$$\mathcal{J} = \left\{ (j_1, \dots, j_R) \in \{0, 1, \dots, J\}^R \mid \sum_{i=1}^R j_i = T \right\}.$$

That is, \mathcal{J} is the set of all nonnegative integer R -tuples which sum to T and with no entry greater than J . Then

$$\sum_{(j_1, \dots, j_R) \in \mathcal{J}} \prod_{i=1}^R \frac{1}{j_i! (J - j_i)!} = \frac{1}{(J!)^R} \binom{JR}{T}.$$

We now give an expression for $M_2(t, a, b, c)$, required for Step 2.

Lemma 5.2. *Suppose that a, b, c, t are nonnegative integers with $0 \leq c \leq t - a - b$, such that if $b = 0$ then $a = 0$ or $a = t$. When $b \geq 1$, the number of templates for F_2 with parameters (a, b, c) is*

$$M_2(t, a, b, c) = \frac{b \xi_t(a, b, c)}{a} \binom{a}{b} \binom{b}{c} \binom{t - a - b}{c}, \quad (5.1)$$

where $\xi_t(a, b, c)$ is defined by

$$\xi_t(a, b, c) = \begin{cases} \frac{c}{t - a - b} & \text{if } a + b < t, \\ 1 & \text{if } a + b = t. \end{cases} \quad (5.2)$$

When $b = 0$ we define $M_2(t, 0, 0, 0) = M_2(t, t, 0, 0) = 1$.

Proof. First, suppose that $1 \leq b \leq t - a - 1$. By Lemma 3.1(ii), there are

$$\binom{b}{c} \binom{t-a-b-1}{c-1} = \frac{c}{t-a-b} \binom{b}{c} \binom{t-a-b}{c}$$

ways to select a sequence $\ell = (\ell_1, \dots, \ell_b)$ of intersection lengths which add to $t - a$, such that precisely $b - c$ of these lengths equal 1 and the rest are at least 2. Then Lemma 3.1(i), there are

$$\binom{a-1}{b-1} = \frac{b}{a} \binom{a}{b}$$

ways to choose a sequence $\mathbf{u} = (u_1, \dots, u_b)$ of gap lengths around H_2 . Multiplying these expressions together gives (5.1).

Next suppose that $b = t - a \geq 0$. The given lower and upper bounds on c imply that $c = 0$. Furthermore, we have $t = a + b \leq 2a$. There is one way to choose the vector ℓ of intersection lengths, and the number of choices for the sequence \mathbf{u} of gap lengths is $\frac{t-a}{a} \binom{a}{t-a}$, as above. This leads to the stated value for $M_2(t, a, t - a, t - a)$, using (5.2) and recalling that $\binom{0}{0} = 1$.

Finally suppose that $b = 0$ (and hence $c = 0$). There are only two possibilities: $a = 0$ and $F_1 = F_2$, corresponding to the unique template $\theta = (1, 1, \dots, 1)$, or $a = t$ and $F_1 \cap F_2 = \emptyset$, corresponding to the unique template $\theta = (0, 0, \dots, 0)$. This matches the values $M_2(t, 0, 0, 0) = M_2(t, t, 0, 0) = 1$. \square

Next we perform Step 5. Recall that during Steps 1–4 we have identified F_1 , the template θ , the parts in $F_1 \cap F_2$ and the connection vertices. In Step 5 we determine the rest of $F_2 \setminus F_1$ and adjust for overcounting, usually by dividing by the number of templates corresponding to a given F_2 .

Lemma 5.3. *Let a, b, t be integers which satisfy $0 \leq b \leq a \leq t$. If $b \geq 1$ then*

$$\begin{aligned} M_5(t, a, b) &= \frac{1}{2b} (a-b)! ((s-2)a)! \left(\frac{(r-1)^{s-2} (r-2)^2}{(s-2)!} \right)^a (r-2)^{-2b} \\ &\quad \times \sum_{d=0}^{a-b} \binom{a-b}{d} \binom{(s-2)a}{a-b-d} \left(\frac{r-3}{r-2} \right)^d, \end{aligned}$$

while if $b = 0$ and $a = t$ then

$$M_5(t, t, 0) = \frac{1}{2t} t! ((s-2)t)! \left(\frac{(r-1)^{s-2} (r-2)^2}{(s-2)!} \right)^t \sum_{d=0}^t \binom{t}{d} \binom{(s-2)t}{t-d} \left(\frac{r-3}{r-2} \right)^d.$$

Finally, $M_5(t, 0, 0) = 1$.

Proof. Let θ be the template for F_2 which was fixed during Step 2. In Step 5, we must identify all H_2 -external vertices in $G(F_2 \setminus F_1)$ which are not connection vertices (there are $a - b$ of them), and all H_2 -internal vertices in $G(F_2 \setminus F_1)$ (there are $(s - 2)a$ of them). As in Lemma 4.2, we call all vertices identified in this step *new*. We must also assign points to all new vertices, thereby completing $F_2 \setminus F_1$. In Section 4 we approximated our number of choices at each step by t or $(s - 2)t$, respectively, since we only had to identify a constant number of new vertices. Here we must count more carefully, and we will need a new parameter.

Let d be the number of new H_2 -external vertices which are also H_1 -external. Then there are $a - b - d$ new H_2 -external vertices which are H_1 -internal, and there are $a - b - d$ new H_2 -internal vertices which are H_2 -external. Finally, there are $(s - 2)a - (a - b - d) = (s - 3)a + b + d$ new H_2 -internal vertices which are H_1 -internal. We must select identities and points for all these new vertices.

To do this, first order all remaining H_1 -external vertices (those not already present in $G(F_1 \cap F_2)$) and order all remaining H_1 -internal vertices (those not already present in $G(F_1 \cap F_2)$), in

$$(a - b)!((s - 2)a)! \quad (5.3)$$

ways. We will work around H_2 in the order specified by the template θ . When we assign identities to new vertices of H_2 , we will always take the first vertex from the appropriate list (either H_1 -external or H_1 -internal), and delete it from this list after it has been assigned. At the end of this process, both lists will be empty and all new vertices in H_2 will have been identified.

To begin, select which d new H_2 -external vertices will be H_1 -external, and assign identities to these vertices in order, from the ordered list of remaining H_1 -external vertices. This can be done in

$$\binom{a - b}{d} \quad (5.4)$$

ways, leaving $a - b - d$ new H_2 -external vertices to be identified. These $a - b - d$ new H_2 -external vertices must be H_1 -internal, and we move around H_2 in order, assigning identities to these vertices from the ordered list of H_1 -internal vertices. (There is only one way to do this.)

Next, by Lemma 5.1, there are

$$\sum_{(j_1, \dots, j_a) \in \mathcal{J}} \prod_{\ell=1}^a \frac{1}{j_\ell! (s - 2 - j_\ell)!} = \frac{1}{((s - 2)!)^a} \binom{(s - 2)a}{a - b - d} \quad (5.5)$$

ways to decide how many new H_2 -internal vertices in each edge of $G(F_2 \setminus F_1)$ will be H_1 -external: let the i th such edge contain j_i H_2 -internal vertices which are H_1 -external. (Here \mathcal{J} is the set of all nonnegative integer sequences of length a which add up to $a - b - d$, such that no entry is greater than $s - 2$.) We visit each edge of $G(F_2 \setminus F_1)$ in order, and in the i th such edge we assign identities to j_i new H_2 -internal vertices

(namely, the first j_i remaining elements from the ordered list of H_1 -external vertices), and we assign identities to the $s - 2 - j_i$ other internal vertices of this edge (from the ordered list of remaining H_1 -internal vertices). We must divide by the factorials to adjust for symmetry, since the H_2 -internal vertices within a new edge should form a set, not a sequence.

This gives every new vertex of H_2 an identity, and now we must assign points. The d new H_2 -external vertices which are H_1 -external and the $a - b - d$ new H_2 -external vertices which are H_1 -internal must all be assigned precisely two points, and all remaining new vertices must be assigned precisely one point. There are

$$\begin{aligned} & ((r-2)(r-3))^d ((r-1)(r-2))^{a-b-d} (r-2)^{a-b-d} (r-1)^{(s-3)a+b+d} \\ &= ((r-1)^{s-2} (r-2)^2)^a (r-2)^{-2b} \left(\frac{r-3}{r-2} \right)^d \end{aligned} \quad (5.6)$$

ways to assign points to these new vertices (in the parts corresponding to the edges of $G(F_2 \setminus F_1)$).

If $b \geq 1$ then F_2 arises from precisely $2b$ templates, so we multiply (5.3)–(5.6) together and divide by $2b$, giving the desired expression. If $b = 0$ and $a = t$ then the above argument determines a sequence of parts of F_2 , with respect to some given start-vertex and direction. We must divide by $2t$ to adjust for this multiple counting. This leads to the stated expression for $M_5(t, t, 0)$. Finally, if $a = b = 0$ then $F_2 = F_1$ and there is nothing to do in this step, so $M_5(t, 0, 0) = 1$ as claimed. \square

Define

$$\mathcal{D} = \{(a, b, c, d) \in \mathbb{Z}^4 \mid 0 \leq c \leq b, \quad 0 \leq d \leq a - b, \quad a + b + c \leq t\}$$

and

$$\widehat{\mathcal{D}} = \mathcal{D} \setminus \{(a, 0, 0, d) \in \mathcal{D} \mid 1 \leq a \leq t - 1\}.$$

The set $\widehat{\mathcal{D}}$ contains all possible 4-tuples of parameters which can arise in the second moment calculation, recalling that when $b = 0$ we must have $a = 0$ or $a = t$, for combinatorial reasons.

The next lemma finds a combinatorial expression for $\mathbb{E}(Y^2)/(\mathbb{E}Y)^2$ as a summation over $\widehat{\mathcal{D}}$, with the summands defined below. However, it will prove easier to calculate the sum over the slightly larger set \mathcal{D} . As we will see, the additional terms will have only negligible effect on the answer. Hence we define the summand $J_t(a, b, c, d)$ for all $(a, b, c, d) \in \mathcal{D}$, as follows. First, let

$$\left. \begin{aligned} \kappa_2 &= \frac{2h(r, s)}{(r-2)^2} \\ \kappa_3 &= \frac{(rs - r - s)^2}{h(r, s)} \\ \kappa_4 &= \frac{r-3}{r-2} \end{aligned} \right\} \quad (5.7)$$

where, as defined in (3.3),

$$h(r, s) = (r - 2)^2 + 2(s - 2)(r - 1)(r - 2) + \frac{1}{2}(s - 2)(s - 3)(r - 1)^2.$$

Lemma 5.4. *Suppose that $s \geq 3$ and $r > \rho(s)$ are fixed integers. Then*

$$\frac{\mathbb{E}(Y^2)}{(\mathbb{E}Y)^2} = \sum_{(a,b,c,d) \in \widehat{\mathcal{D}}} J_t(a, b, c, d)$$

where the summands are defined as follows:

- If $a = 0$ then $b = c = d = 0$ and we define $J_t(0, 0, 0, 0) = \frac{1}{\mathbb{E}Y}$.
- If $a \geq 1$ then we let

$$\begin{aligned} J_t(a, b, c, d) &= \frac{\xi_t(a, b, c) t}{2a^2} \binom{a}{b} \binom{b}{c} \binom{t-a-b}{c} a! ((s-2)a)! \\ &\quad \times \binom{a-b}{d} \binom{(s-2)a}{a-b-d} \left(\frac{(r-1)^{s-2} (r-2)^2}{(s-2)!} \right)^a \kappa_2^b \kappa_3^c \kappa_4^d \\ &\quad \times \frac{p(rn - s(t+a))}{p(rn - st)} \frac{1}{\mathbb{E}Y}, \end{aligned}$$

where $\xi_t(a, b, c)$ is defined in (5.2).

Proof. The set $\widehat{\mathcal{D}}$ contains all values of the parameters (a, b, c, d) which can arise from the interaction of two loose Hamilton cycles. After dividing (3.4) by $\mathbb{E}Y$, we can write the resulting expression as a sum over $\widehat{\mathcal{D}}$, and denote the summand corresponding to $(a, b, c, d) \in \widehat{\mathcal{D}}$ by $J_t(a, b, c, d)$. (Recall that the sum over d arises in the factor $M_5(t, a, b)$, see Lemma 5.3.)

When $b \geq 1$, substituting Lemma 5.2 and Lemma 5.3 into (3.4) and dividing by $\mathbb{E}Y$ shows that the summand $J_t(a, b, c, d)$ equals the expression given in above.

When $a = 0$ we have $b = c = d = 0$, corresponding to the term

$$\frac{M_5(t, 0, 0)}{\mathbb{E}Y} = \frac{1}{\mathbb{E}Y}.$$

This equals the definition of $J_t(0, 0, 0, 0)$ given above. Finally, suppose that $a = t$ and $b = c = 0$, which corresponds to

$$\begin{aligned} \frac{M_5(t, t, 0)}{\mathbb{E}Y} \cdot \frac{p(rn - 2st)}{p(rn - st)} &= \frac{t! ((s-2)t)!}{2t} \left(\frac{(r-1)^{s-2} (r-2)^2}{(s-2)!} \right)^t \sum_{d=0}^t \binom{t}{d} \binom{(s-2)t}{t-d} \\ &\quad \times \left(\frac{r-3}{r-2} \right)^d \frac{p(rn - 2st)}{p(rn - st)} \frac{1}{\mathbb{E}Y}. \end{aligned}$$

Finally, observe that this expression equals $\sum_{d=0}^t J_t(t, 0, 0, d)$, with $J_t(t, 0, 0, d)$ as defined above, noting that $\xi_t(t, 0, 0) = 1$. \square

The summation in Lemma 5.4 will be evaluated using Laplace summation. The following lemma is tailored for this purpose: it is a restatement of [9, Lemma 6.3] (using the notation of the current paper).

Lemma 5.5. *Suppose the following:*

- (i) $\mathcal{L} \subset \mathbb{R}^m$ is a lattice with full rank m .
- (ii) $K \subset \mathbb{R}^m$ is a compact convex set with non-empty interior K° .
- (iii) $\varphi : K \rightarrow \mathbb{R}$ is a continuous function with a unique maximum at some interior point $\mathbf{x}^* \in K^\circ$.
- (iv) φ is twice continuously differentiable in a neighbourhood of \mathbf{x}^* and the Hessian $H^* := D^2\varphi(\mathbf{x}^*)$ is strictly negative definite.
- (v) $\psi : K^* \rightarrow \mathbb{R}$ is a continuous function on some neighbourhood $K^* \subseteq K$ of \mathbf{x}^* with $\psi(\mathbf{x}^*) > 0$.
- (vi) For each positive integer t there is a vector $\mathbf{w}_t \in \mathbb{R}^m$.
- (vii) For each positive integer t there is a function $J_t : (\mathcal{L} + \mathbf{w}_t) \cap tK \rightarrow \mathbb{R}$ and a real number $b_t > 0$ such that, as $t \rightarrow \infty$,

$$J_t(\mathbf{v}) = O\left(b_t e^{t\varphi(\mathbf{v}/t) + o(t)}\right), \quad \mathbf{v} \in (\mathcal{L} + \mathbf{w}_t) \cap tK, \quad (5.8)$$

and

$$J_t(\mathbf{v}) = b_t (\psi(\mathbf{v}/t) + o(1)) e^{t\varphi(\mathbf{v}/t)}, \quad \mathbf{v} \in (\mathcal{L} + \mathbf{w}_t) \cap tK^*, \quad (5.9)$$

uniformly for \mathbf{v} in the indicated sets.

Then, as $t \rightarrow \infty$,

$$\sum_{\mathbf{v} \in (\mathcal{L} + \mathbf{w}_t) \cap tK} J_t(\mathbf{v}) \sim \frac{(2\pi)^{m/2} \psi(\mathbf{x}^*)}{\det(\mathcal{L}) \sqrt{\det(-H^*)}} b_t t^{m/2} e^{t\varphi(\mathbf{x}^*)}. \quad (5.10)$$

In order to apply Lemma 5.5, we need some more notation. Define the scaled domain

$$K = \{(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \mid 0 \leq \gamma \leq \beta, \quad 0 \leq \delta \leq \alpha - \beta, \quad \alpha + \beta + \gamma \leq 1\}. \quad (5.11)$$

Observe that \mathcal{D} can be written as the intersection of \mathbb{Z}^4 with tK , but it is not possible to write $\widehat{\mathcal{D}}$ in this form. This is the reason why it is more convenient to work with \mathcal{D} when performing Laplace summation.

Let

$$\kappa_1 = (r-1)^{s-2} (r-2)^2 (s-1)(s-2)^{2(s-2)}$$

and recall (5.7). Define the function $\varphi : K \rightarrow \mathbb{R}$ by

$$\begin{aligned} \varphi(\alpha, \beta, \gamma, \delta) = & g(1 - \alpha - \beta) + 2(s-1)g(\alpha) + \frac{s-1}{s}g(rs - r - s - s\alpha) - g(\beta - \gamma) \\ & - g(\delta) - 2g(\gamma) - 2g(\alpha - \beta - \delta) - g((s-3)\alpha + \beta + \delta) \\ & - g(1 - \alpha - \beta - \gamma) + \alpha \ln(\kappa_1) + \beta \ln(\kappa_2) + \gamma \ln(\kappa_3) + \delta \ln(\kappa_4) \end{aligned} \quad (5.12)$$

where $g(x) = x \ln x$ and $g(0) = 0$. We will need the following information about the function φ . The proof of the following crucial result is lengthy and technical, so it is deferred to the appendix.

Lemma 5.6. *Suppose that $s \geq 3$ and $r > \rho(s)$, where $\rho(s)$ is defined in Theorem 1.1. Then φ has a unique global maximum over the domain K which occurs at the point $\mathbf{x}^* = (\alpha^*, \beta^*, \gamma^*, \delta^*)$ defined by*

$$\begin{aligned} \alpha^* &= \frac{rs - r - s}{r(s-1)}, & \beta^* &= \frac{rs - s - 2}{r(r-1)(s-1)}, \\ \gamma^* &= \frac{2(rs - r - s)}{r(r-1)^2(s-1)^2}, & \delta^* &= \frac{(r-2)(r-3)}{r(r-1)(s-1)}. \end{aligned}$$

The maximum value of φ equals

$$\varphi(\mathbf{x}^*) = \ln(r-1) + \ln(s-1) + \frac{(s-1)(rs - r - s)}{s} \ln \left(\frac{(rs - r - s)^2}{rs - r} \right). \quad (5.13)$$

Let $Q(r, s)$ be as defined in Lemma 4.6, and denote by H^* the Hessian of $\varphi(\alpha, \beta, \gamma, \delta)$ evaluated at the point \mathbf{x}^* . Then H^* is strictly negative definite and

$$\det(-H^*) = \frac{r^5(r-1)^5(r-2)(s-1)^8 Q(r, s)}{8(r-3)(rs - r - s)^2 (rs - 2r - 2s + 4)^2 h(r, s)}. \quad (5.14)$$

With this result in hand, we may establish the following asymptotic expression for the second moment of Y .

Lemma 5.7. *Suppose that $s \geq 3$ and $r > \rho(s)$ are fixed integers, where $\rho(s)$ is defined in Theorem 1.1. Then*

$$\frac{\mathbb{E}(Y^2)}{(\mathbb{E}Y)^2} \sim \frac{r(rs - r - s)}{(r-2)\sqrt{Q(r, s)}}$$

where $Q(r, s)$ is defined in Lemma 4.6.

Proof. Firstly, extend the definition of $J_t(a, b, c, d)$ to cover all $(a, b, c, d) \in \mathcal{D}$, by defining $\xi_t(a, b, c) = 1$ if $b = c = 0$ and $1 \leq a \leq t - 1$. We will apply Lemma 5.5 to calculate the sum of $J_t(a, b, c, d)$ over the domain \mathcal{D} . While doing so, we will observe that the contribution to the sum from $\mathcal{D} \setminus \widehat{\mathcal{D}}$ is negligible, which will imply that the sum over the larger domain \mathcal{D} is also asymptotically equal to $\mathbb{E}(Y^2)/(\mathbb{E}Y)$.

The first six conditions of Lemma 5.5 hold with the definitions given below.

- (i) Let $\mathcal{L} = \mathbb{Z}^4$, a lattice with full rank 4 and with determinant 1.
- (ii) The domain K defined in (5.11) is compact, convex, is contained in $[0, 1]^4$ and has non-empty interior. Observe that $\mathcal{D} = \mathbb{Z}^4 \cap tK$.
- (iii) The function $\varphi : K \rightarrow \mathbb{R}$ defined before Lemma 5.6 is continuous. Furthermore, φ has a unique maximum at the point \mathbf{x}^* , by Lemma 5.6, and \mathbf{x}^* belongs to the interior of K .
- (iv) The function φ is infinitely differentiable in the interior of K . Let H^* denote the Hessian matrix of φ evaluated at \mathbf{x}^* . Then H^* is strictly negative definite, by Lemma 5.6.
- (v) Write $\varepsilon = (r(r-1)^2(s-1)^2)^{-1}$ and define

$$K^* = \{(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \mid \varepsilon < \gamma \leq \beta - \varepsilon, \quad \varepsilon < \delta < \alpha - \beta - \varepsilon, \quad \alpha + \beta + \gamma \leq 1 - \varepsilon\}.$$

Then K^* is contained in the interior of K and $\mathbf{x}^* \in K^*$. Furthermore, the function $\psi : K^* \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \psi(\alpha, \beta, \gamma, \delta) \\ = \frac{1}{(\alpha - \beta - \delta) \sqrt{(\beta - \gamma) \delta (1 - \alpha - \beta) (1 - \alpha - \beta - \gamma) ((s-3)\alpha + \beta + \delta)}} \end{aligned}$$

is continuous on K^* . Direct substitution shows that

$$\psi(\mathbf{x}^*) = \frac{1}{2(s-2)(rs-2r-2s+4)} \sqrt{\frac{r^7 (r-1)^5 (s-1)^9}{2(r-2)(r-3)h(r,s)}} \quad (5.15)$$

which is certainly positive when $r > \rho(s)$ and $s \geq 3$.

- (vi) Let \mathbf{w}_t be the zero vector for each t .

It remains to prove that condition (vii) of Lemma 5.5 holds. Define

$$b_t = \frac{(s-2)}{4\pi^2 t^2 \sqrt{s-1}} \left(\frac{1}{(r-1)(s-1)} \left(\frac{rs-r}{(rs-r-s)^2} \right)^{(s-1)(rs-r-s)/s} \right)^t, \quad (5.16)$$

and introduce the scaled variables

$$\alpha = a/t, \quad \beta = b/t, \quad \gamma = c/t, \quad \delta = d/t. \quad (5.17)$$

Observe that (5.8) holds when $a = 0$, since then $\varphi(0, 0, 0, 0) = \frac{s-1}{s}g(rs - r - s)$ and

$$J_t(0, 0, 0, 0) = \frac{1}{\mathbb{E}Y} = O(b_t) \exp\left(t\varphi(0, 0, 0, 0) + \frac{3}{2}\ln t\right),$$

using Corollary 2.2. When $a \geq 1$, first rewrite all binomial coefficients in $J_t(a, b, c, d)$ in terms of factorials (except those in the factor $\mathbb{E}Y$), giving

$$\begin{aligned} J_t(a, b, c, d) &= \frac{\xi_t(a, b, c) t}{2a^2} (s(s-1)(r-1)^{s-2}(r-2)^2)^a \kappa_2^b \kappa_3^c \kappa_4^d \\ &\quad \times \frac{(a!)^2 (t-a-b)! (((s-2)a)!)^2}{(b-c)! d! (c!)^2 ((a-b-d)!)^2 (t-a-b-c)! ((s-3)a+b+d)!} \\ &\quad \times \frac{(rn-s(t+a))! (rn/s-t)!}{(rn/s-(t+a))! (rn-st)!} \frac{1}{\mathbb{E}Y}. \end{aligned}$$

Let $x \vee y$ denote $\max(x, y)$ and apply Stirling's formula in the form

$$\ln(N!) = N \ln N - N + \frac{1}{2} \ln(N \vee 1) + \frac{1}{2} \ln 2\pi + O(1/(N+1)),$$

valid for all integers $N \geq 0$, to the above expression for $J_t(a, b, c, d)$. After substituting for $\mathbb{E}Y$ using Corollary 2.2, this gives

$$\begin{aligned} J_t(a, b, c, d) &= \left(1 + O\left(\frac{1}{c+1} + \frac{1}{d+1} + \frac{1}{b-c+1} + \frac{1}{a-b-d+1} + \frac{1}{t-a-b-c+1}\right)\right) \\ &\quad \times \frac{(s-2)t^{3/2} (\kappa_1 t^{-(s-1)})^a \kappa_2^b \kappa_3^c \kappa_4^d}{4\pi^2 (a-b-d) \sqrt{(s-1)(b-c)d(t-a-b)(t-a-b-c)((s-3)a+b+d)}} \\ &\quad \times \frac{(t-a-b)^{t-a-b} a^{2(s-1)a} (rs-r-s-sa/t)^{(s-1)(rs-r-s-sa/t)t/s}}{(b-c)^{b-c} c^{2c} d^d (a-b-d)^{2(a-b-d)} (t-a-b-c)^{t-a-b-c} ((s-3)a+b+d)^{(s-3)a+b+d}} \\ &\quad \times \left(\frac{1}{(r-1)(s-1)} \left(\frac{rs-r}{(rs-r-s)^2}\right)^{(s-1)(rs-r-s)/s}\right)^t, \end{aligned}$$

except that if some factor in the denominator is zero then it should be replaced by 1. (Also interpret 0^0 as 1.) Finally, rewriting this expression in terms of the scaled variables (5.17) proves that (5.8) and (5.9) hold. Hence condition (vii) of Lemma 5.5 is satisfied.

Therefore we may apply Lemma 5.5 to conclude that (5.10) holds: that is,

$$\sum_{(a,b,c,d) \in \mathcal{D}} J_t(a, b, c, d) \sim \frac{4\pi^2 \psi(\mathbf{x}^*)}{\sqrt{\det(-H^*)}} b_t t^2 e^{t\varphi(\mathbf{x}^*)}, \quad (5.18)$$

using the fact that the lattice \mathbb{Z}^4 has determinant 1. Observe that up to the $1 + o(1)$ relative error term, the answer depends only on $\mathcal{D} \cap tK^*$, which equals $\widehat{\mathcal{D}} \cap tK^*$. (Indeed, the only terms which contribute non-negligibly to the sum are those which are close to \mathbf{x}^* .) Therefore we may replace \mathcal{D} by $\widehat{\mathcal{D}}$ in (5.18). By Lemma 5.4, the proof is completed by substituting (5.13) – (5.16) into (5.18). \square

6 Proof of the threshold result

To prepare for the proof of Theorem 1.1, we now find the values of r, s for which $\mathbb{E}Y$ tends to infinity. Define

$$L(r, s) = \ln(r - 1) + \ln(s - 1) + \frac{(s - 1)(rs - r - s)}{s} \ln \left(1 - \frac{s}{rs - r} \right)$$

and treat r as a continuous variable. Then $L(r, s)$ is the natural logarithm of the base of the exponential factor in $\mathbb{E}Y$, see Corollary 2.2. If $L(r, s) \leq 0$ then $\mathbb{E}(Y) = o(1)$, so a.a.s. there are no loose Hamilton cycles. For example, $L(3, 3) = 0$, so a.a.s. $F \in \mathcal{F}(n, 3, 3)$ has no loose Hamilton cycles. Similarly, if $L(r, s) > 0$ then $\mathbb{E}(Y) \rightarrow \infty$.

Lemma 6.1. *For any fixed integer $s \geq 2$, there exists a unique real number $\rho(s) > 2$ satisfying the lower and upper bounds given in (1.3) such that $L(\rho(s), s) = 0$,*

$$L(r, s) < 0 \quad \text{for } r \in [2, \rho(s)) \quad \text{and} \quad L(r, s) > 0 \quad \text{for } r \in (\rho(s), \infty).$$

Proof. The statements hold when $s \in \{2, 3\}$, as can be verified directly. For the remainder of the proof, assume that $s \geq 4$. Setting $x = (rs - r - s)/s$, we may rewrite $L(r, s)$ as $L(r, s) = f_s(x)$ where

$$f_s(x) = \ln(sx + 1) - (s - 1)x \ln \left(1 + \frac{1}{x} \right).$$

First we claim that $f_s(x)$ is negative when $x \in [1 - \frac{2}{s}, s - 1 - \frac{1}{s}]$. (This range of x corresponds to $2 \leq r \leq s + 1$.) This can be verified directly for $s = 4, \dots, 13$, while for fixed $s \geq 14$ we have, for x in this range,

$$\begin{aligned} f_s(x) &\leq \ln(sx + 1) - (s - 1)x \left(\frac{1}{x} - \frac{1}{2x^2} \right) \\ &\leq \ln(s(s - 1)) - (s - 1) + \frac{s(s - 1)}{2(s - 2)} < 0. \end{aligned}$$

This implies that $L(r, s) < 0$ for all $s \geq 4$ and $2 \leq r \leq s + 1$.

Next, suppose that $x \geq s - 1 - \frac{1}{s}$. Then the derivative of f_s with respect to x satisfies

$$\begin{aligned} f'_s(x) &= \frac{s}{sx+1} - (s-1) \ln\left(1 + \frac{1}{x}\right) + \frac{s-1}{x+1} \geq \frac{s}{sx+1} - \frac{s-1}{x} + \frac{s-1}{x+1} \\ &= \frac{sx(x-s) + 2sx - (s-1)}{(sx+1)x(x+1)} \geq \frac{sx(-1-1/s) + 2sx - (s-1)}{(sx+1)x(x+1)} > 0. \end{aligned}$$

This implies that $L(r, s)$ is monotonically increasing as a function of $r \geq s+1$, for any fixed $s \geq 4$, as

$$\frac{\partial}{\partial r} L(r, s) = \frac{(s-1)}{s} f'_s(x).$$

Furthermore, $f_s(x)$ tends to infinity as $x \rightarrow \infty$. Therefore the function $L(\cdot, s)$ has precisely one root in $(2, \infty)$, for all $s \geq 4$. Let this root be $r = \rho(s)$. Next we will prove that

$$f_s(x_1) < 0 \quad \text{and} \quad f_s(x_2) > 0 \tag{6.1}$$

where

$$x_1 = \frac{e^{s-1}}{s} - \frac{s-1}{2} - \frac{1}{s} - \frac{(s^2 - s + 1)^2}{se^{s-1}}, \quad x_2 = \frac{e^{s-1}}{s} - \frac{s-1}{2} - \frac{1}{s}.$$

Since $\rho^-(s) = s(x_1 + 1)/(s-1)$ and $\rho^+(s) = s(x_2 + 1)/(s-1)$, this will prove that $\rho^-(s) < \rho(s) < \rho^+(s)$, as required.

When $s = 4, 5$, we can verify the inequalities (6.1) directly. Now suppose that $s \geq 6$. Using the inequality $a \ln(1 + 1/a) \geq 1 - \frac{1}{2a}$, which holds for all $a > 1$, we have

$$\begin{aligned} \exp\left((s-1)x \ln\left(1 + \frac{1}{x}\right)\right) &\geq \exp\left(s - 1 - \frac{s-1}{2x}\right) \\ &\geq e^{s-1} \left(1 - \frac{s-1}{2x}\right). \end{aligned}$$

Note that $f_s(x) < 0$ if this expression is bounded below by $sx + 1$. This holds if and only if

$$2sx^2 - 2(e^{s-1} - 1)x + (s-1)e^{s-1} < 0.$$

Let x^- and x^+ denote the smaller and larger root of this quadratic (in x), respectively. Then

$$\begin{aligned} x^+ &= \frac{e^{s-1} - 1}{2s} + \frac{e^{s-1}}{2s} \sqrt{1 - \frac{2(s^2 - s + 1)}{e^{s-1}}} + \frac{1}{e^{2(s-1)}} \\ &\geq \frac{e^{s-1} - 1}{2s} + \frac{e^{s-1}}{2s} \sqrt{1 - \frac{2(s^2 - s + 1)}{e^{s-1}}} \geq x_1 \end{aligned}$$

using the inequality $(1 - a)^{1/2} \geq 1 - a/2 - a^2/2$, which holds for all $a \in (0, 1)$. Also

$$x^- < \frac{e^{s-1} - 1}{2s} < x_1,$$

proving the first statement in (6.1).

For the upper bound, by definition of x_2 , we have

$$\ln(sx_2 + 1) = s - 1 + \ln\left(1 - \frac{s(s-1)}{2e^{s-1}}\right) \geq (s-1)\left(1 - \frac{s}{2e^{s-1}} - \frac{s^2(s-1)}{6e^{2(s-1)}}\right)$$

since $\frac{s(s-1)}{2e^{s-1}} < 1/3$ when $s \geq 6$, and $\ln(1-a) \geq -a - 2a^2/3$ when $0 < a < 1/3$. Next, since $x_2 > 1$ we have

$$(s-1)x_2 \ln\left(1 + \frac{1}{x_2}\right) \leq (s-1)\left(1 - \frac{1}{2x_2} + \frac{1}{3x_2^2}\right).$$

Hence $f(x_2) > 0$ holds if

$$\frac{1}{3x_2^2} < \frac{1}{2x_2} - \frac{s}{2e^{s-1}} - \frac{s^2(s-1)}{6e^{2(s-1)}}. \quad (6.2)$$

Substituting the expression for x_2 , the left hand side of (6.2) becomes

$$\frac{4s^2}{3(2e^{s-1} - (s^2 - s + 2))^2}$$

while the right hand side of (6.2) becomes

$$\begin{aligned} \frac{s(s^2 - s + 2)}{2e^{s-1}(2e^{s-1} - (s^2 - s + 2))} - \frac{s^2(s-1)}{6e^{2(s-1)}} &> \frac{s(s^2 - s + 2)}{4e^{2(s-1)}} - \frac{s^2(s-1)}{6e^{2(s-1)}} \\ &= \frac{s(s^2 - s + 6)}{12e^{2(s-1)}}. \end{aligned}$$

Therefore, it suffices to prove that

$$16s^2 e^{2(s-1)} \leq s(s^2 - s + 6) (2e^{s-1} - (s^2 - s + 2))^2.$$

But the right hand side of this expression is bounded below by $4 \cdot \frac{49}{64} s(s^2 - s + 6) e^{2(s-1)}$ when $s \geq 6$. For $s \geq 6$ the inequality $256s \leq 49(s^2 - s + 6)$ holds, and hence (6.2) holds. Therefore $f_s(x_2) > 0$ for $s \geq 6$, completing the proof. \square

We may now complete the proof of our main result, Theorem 1.1, establishing a threshold result for existence of a loose Hamilton cycle in $\mathcal{G}(n, r, s)$.

Proof of Theorem 1.1. When $s = 2$, the result follows immediately from Robinson and Wormald [12, 13], since $\rho(2) < 3$. Now suppose that $s \geq 3$. Lemma 6.1 proves that there is a unique value of $\rho(s) \geq 3$ such that

$$(r-1)(s-1) \left(\frac{rs - r - s}{rs - r} \right)^{(s-1)(rs-r-s)/s} = 1.$$

Furthermore, Lemma 6.1 proved that the upper and lower bounds on $\rho(s)$ given in (1.3) hold.

If $r \geq 3$ is an integer with $r \leq \rho(s)$ then a.a.s. $G \in \mathcal{G}(n, r, s)$ contains no loose Hamilton cycle, using (2.5) and Lemma 2.2, since $\Pr(Y) \leq \mathbb{E}(Y)$. (This can also be deduced from Theorem 1.2.)

Now suppose that $r > \rho(s)$ for some fixed $s \geq 3$. As noted in Section 1, Cooper et al. [2] proved that condition (A1) of Theorem 2.3 holds, with λ_k as defined in (2.8). The remaining conditions of Theorem 2.3 have also been established: Corollary 4.4 proves that condition (A2) holds, Lemma 4.6 shows that condition (A3) holds, while condition (A4) is shown to hold by combining Corollary 4.4 and Lemma 5.7. Also recall that $\delta_k > -1$ for all $k \geq 1$, by Lemma 4.5. Therefore Lemma 2.5 implies that a.a.s. $Y_G > 0$. This shows that a.a.s. $G \in \mathcal{G}(n, r, s)$ contains a loose Hamilton cycle whenever $s \geq 3$ and $r > \rho(s)$, completing the proof. \square

Finally, we provide the asymptotic distribution of the number of loose Hamilton cycles in $\mathcal{G}(n, r, s)$.

Theorem 6.2. *Suppose that r, s are integers with $s \geq 2$ and $r > \rho(s)$. Recall that Y_G is the number of loose Hamilton cycles in $\mathcal{G}(n, r, s)$. Then with $\zeta_1, \zeta_2 \in \mathbb{C}$ satisfying (4.3),*

$$\frac{Y_G}{\mathbb{E}(Y_G)} \xrightarrow{d} \prod_{k=2}^{\infty} \left(1 + \frac{\zeta_1^k + \zeta_2^k - 1}{((r-1)(s-1))^k} \right)^{Z_k} \exp \left(\frac{1 - \zeta_1^k - \zeta_2^k}{2k} \right)$$

where the variables Z_k are independent Poisson variables with

$$\mathbb{E}Z_k = \frac{((r-1)(s-1))^k}{2k}$$

for $k \geq 2$.

Proof. The case $s = 2$ was proved by Janson [11, Theorem 2]. (To see that our expression matches his, observe that when $s = 2$ we have $\{\zeta_1, \zeta_2\} = \{0, -1\}$. Hence the factor of W corresponding to any even value of k equals 1.) For $s \geq 3$, we showed in the proof of Theorem 1.1 that conditions (A1)–(A4) of Theorem 2.3 hold for Y . Then the result follows by Lemma 2.5. \square

References

- [1] P. Allen, J. Böttcher, Y. Kohayakawa and Y. Person, Tight Hamilton cycles in random hypergraphs, *Random Structures and Algorithms* **46** (2015), 446–465.
- [2] C. Cooper, A. Frieze, M. Molloy and B. Reed, Perfect matchings in random r -regular, s -uniform hypergraphs, *Electronic Journal of Combinatorics* **7** (2000), #R57.

- [3] P. Duchet, Hypergraphs, in *Handbook of Combinatorics* **1**, MIT Press, Cambridge, Massachusetts, pp. 381–432, 1995.
- [4] A. Dudek and A. Frieze, Tight Hamilton cycles in random uniform hypergraphs, *Random Structures and Algorithms* **42** (2013), 374–375.
- [5] A. Dudek, A. Frieze, A. Ruciński and M. Šileikis, Approximate counting of regular hypergraphs, *Information Processing Letters* **113** (2013), 785–788.
- [6] A. Dudek, A. Frieze, A. Ruciński and M. Šileikis, Loose Hamilton cycles in regular hypergraphs, *Combinatorics, Probability and Computing* **24** (2015), 179–194.
- [7] A. Ferber, Closing gaps in problems related to Hamilton cycles in random graphs and hypergraphs, *Electronic Journal of Combinatorics* **22(1)** (2015), #P1.61.
- [8] A. Frieze, M. Jerrum, M. Molloy, R. W. Robinson, and N.C. Wormald, Generating and counting Hamilton cycles in random regular graphs, *Journal of Algorithms* **21** (1996), 176–198.
- [9] C. Greenhill, S. Janson, and A. Ruciński, On the number of perfect matchings in random lifts, *Combinatorics, Probability and Computing* **19** (2010), 791 – 817.
- [10] M. Isaev and B.D. McKay, Complex martingales and asymptotic enumeration, Preprint, 2016. [arXiv:1604.08305](https://arxiv.org/abs/1604.08305)
- [11] S. Janson, Random regular graphs: asymptotic distributions and contiguity, *Combinatorics, Probability and Computing* **4** (1995), 369–405.
- [12] R. W. Robinson and N.C. Wormald, Almost all cubic graphs are hamiltonian, *Random Structures and Algorithms* **3** (1992), 117–125.
- [13] R. W. Robinson and N.C. Wormald, Almost all regular graphs are hamiltonian, *Random Structures and Algorithms* **5** (1994), 363–374.
- [14] N.C. Wormald, Models of random regular graphs, in *Surveys in Combinatorics, 1999* (J.D. Lamb and D.A. Preece, eds), London Mathematical Society Lecture Note Series **267**, Cambridge University Press, Cambridge, pp. 239–298, 1999.

A Search for the global maximum

We now present the proof of Lemma 5.6.

Observe that K , defined in (5.11), is a compact, convex set in $[0, 1]^4$. Furthermore, φ , defined in (5.12), is a continuous function on K . Therefore φ attains its maximum

value at least once. Moreover, φ is infinitely differentiable in the interior of K . The first-order partial derivatives of φ are given by

$$\begin{aligned} \frac{\partial \varphi}{\partial \alpha} = & -\ln(1 - \alpha - \beta) + 2(s-1)\ln(\alpha) - 2\ln(\alpha - \beta - \delta) + \ln(1 - \alpha - \beta - \gamma) \\ & - (s-3)\ln((s-3)\alpha + \beta + \delta) - (s-1)\ln(rs - r - s - s\alpha) + \ln(\kappa_1), \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} \frac{\partial \varphi}{\partial \beta} = & -\ln(1 - \alpha - \beta) + 2\ln(\alpha - \beta - \delta) - \ln(\beta - \gamma) + \ln(1 - \alpha - \beta - \gamma) \\ & - \ln((s-3)\alpha + \beta + \delta) + \ln(\kappa_2), \end{aligned} \quad (\text{A.2})$$

$$\frac{\partial \varphi}{\partial \gamma} = \ln(\beta - \gamma) - 2\ln(\gamma) + \ln(1 - \alpha - \beta - \gamma) + \ln(\kappa_3), \quad (\text{A.3})$$

$$\frac{\partial \varphi}{\partial \delta} = -\ln(\delta) + 2\ln(\alpha - \beta - \delta) - \ln((s-3)\alpha + \beta + \delta) + \ln(\kappa_4). \quad (\text{A.4})$$

For $\tau \geq 1$, let

$$p_\tau = \frac{(\tau-1)^2 + \kappa_3(\tau-1)}{(\tau-1)^2 + 2\kappa_3\tau - \kappa_3}, \quad q_\tau = \frac{\kappa_4\tau(\tau-1)^2}{\kappa_2((\tau-1)^2 + 2\kappa_3\tau - \kappa_3)}$$

and define $\mathbf{x}_\tau = (\alpha_\tau, \beta_\tau, \gamma_\tau, \delta_\tau)$ by

$$\begin{aligned} \alpha_\tau = & \frac{p_\tau + q_\tau + \frac{(s-3)q_\tau}{2\kappa_4} + \sqrt{\frac{(s-2)q_\tau(p_\tau+q_\tau)}{\kappa_4} + \frac{(s-3)^2q_\tau^2}{4\kappa_4^2}}}{1 + p_\tau + q_\tau + \frac{(s-3)q_\tau}{2\kappa_4} + \sqrt{\frac{(s-2)q_\tau(p_\tau+q_\tau)}{\kappa_4} + \frac{(s-3)^2q_\tau^2}{4\kappa_4^2}}}, \\ \beta_\tau = & p_\tau(1 - \alpha_\tau), \quad \gamma_\tau = \frac{\kappa_3(\tau-1)}{(\tau-1)^2 + 2\kappa_3\tau - \kappa_3}(1 - \alpha_\tau), \quad \delta_\tau = q_\tau(1 - \alpha_\tau). \end{aligned}$$

Since $\kappa_2, \kappa_3, \kappa_4 > 0$ it follows immediately that $\alpha_1 = 0$ and $\lim_{\tau \rightarrow \infty} \alpha_\tau = 1$. Next, observe that both p_τ and q_τ/p_τ are increasing, infinitely differentiable functions of $\tau \in [1, \infty)$, and hence

$$\alpha_\tau \text{ is a strictly increasing differentiable function of } \tau \in [1, \infty). \quad (\text{A.5})$$

Observe also that

$$\mathbf{x}_\tau \text{ lies in the interior of the domain } K, \text{ for any } \tau > 1. \quad (\text{A.6})$$

Recalling that $\kappa_4 < 1$, we have

$$\sqrt{\frac{(s-2)q_\tau(p_\tau+q_\tau)}{\kappa_4} + \frac{(s-3)^2q_\tau^2}{4\kappa_4^2}} \geq q_\tau + \frac{(s-3)q_\tau}{2\kappa_4}, \quad (\text{A.7})$$

and hence

$$(1 - \alpha_\tau)^{-1} \geq 1 + p_\tau + \left(2 + \frac{s-3}{\kappa_4}\right) q_\tau \geq 1 + p_\tau + (s-1)q_\tau. \quad (\text{A.8})$$

We also note that $\tau = \frac{1-\alpha_\tau-\beta_\tau}{1-\alpha_\tau-\beta_\tau-\gamma_\tau}$ for all $\tau \geq 1$.

Our interest in this particular curve \mathbf{x}_τ is clarified by the following lemma, which shows that \mathbf{x}_τ is a parameterisation of a ridge which must contain any global maximum of φ . We prove Lemma A.1 in Section A.1.

Lemma A.1. *Suppose that the assumptions of Lemma 5.6 hold. For $\tau \geq 1$, the function $\eta_\tau(\beta, \gamma, \delta) = \varphi(\alpha_\tau, \beta, \gamma, \delta)$ has a unique global maximum over the domain*

$$K_\tau = \{(\beta, \gamma, \delta) \in \mathbb{R}^3 \mid 0 \leq \gamma \leq \beta, \ 0 \leq \delta \leq \alpha_\tau - \beta, \ \beta + \gamma \leq 1 - \alpha_\tau\}$$

which occurs at the point $(\beta_\tau, \gamma_\tau, \delta_\tau)$. In particular, $\frac{\partial}{\partial \beta} \varphi(\mathbf{x}_\tau) = \frac{\partial}{\partial \gamma} \varphi(\mathbf{x}_\tau) = \frac{\partial}{\partial \delta} \varphi(\mathbf{x}_\tau) = 0$ for any $\tau > 1$.

Recall the point \mathbf{x}^* from the statement of Lemma 5.6. Note that \mathbf{x}^* belongs to the ridge, namely, $\mathbf{x}^* = \mathbf{x}_{\tau^*}$ for $\tau^* = (r-1)(s-1)$. Therefore, by (A.6), the point \mathbf{x}^* belongs to the interior of K . Direct substitution shows that (5.13) and (5.14) hold, and also that $\frac{\partial}{\partial \alpha} \varphi(\mathbf{x}^*) = 0$. Therefore \mathbf{x}^* is a stationary point of φ with $\det(-H^*) > 0$. Lemma A.1 implies that all eigenvalues of $-H^*$ are strictly positive, since otherwise there should be at least two negative eigenvalues and then $(\beta^*, \gamma^*, \delta^*)$ could not maximize η_{τ^*} . In particular, this shows that $-H^*$ is positive definite, and hence H^* is negative definite, as claimed. Therefore, \mathbf{x}^* is a local maximum of φ . It remains to prove that $\varphi(\mathbf{x}^*)$ is strictly larger than $\varphi(\mathbf{x}_\tau)$ for all $\tau \geq 1$ with $\tau \neq \tau^*$.

First we consider $\tau = 1$. Note that $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, therefore

$$\varphi(\mathbf{x}_1) = \frac{(s-1)(rs-r-s)}{s} \ln(rs-r-s).$$

Using (5.13), we find that

$$\varphi(\mathbf{x}^*) - \varphi(\mathbf{x}_1) = \ln(r-1) + \ln(s-1) + \frac{(s-1)(rs-r-s)}{s} \ln\left(\frac{rs-r-s}{rs-r}\right), \quad (\text{A.9})$$

and this expression is positive since $r > \rho(s)$ (see Lemma 6.1).

Now we may suppose that $\tau > 1$. Let φ_α denote the partial derivative of φ with respect to α . By Lemma A.1 we have $\frac{\partial}{\partial \delta} \varphi(\mathbf{x}_\tau) = 0$. Combining (A.1) and (A.4), we find that

$$\begin{aligned} \varphi_\alpha(\mathbf{x}_\tau) &= -\ln(1-\alpha_\tau-\beta_\tau) + 2(s-1)\ln\alpha_\tau - 2\ln(\alpha_\tau-\beta_\tau-\delta_\tau) + \ln(1-\alpha_\tau-\beta_\tau-\gamma_\tau) \\ &\quad - (s-3)\ln((s-3)\alpha_\tau+\beta_\tau+\delta_\tau) - (s-1)\ln(rs-r-s-s\alpha_\tau) + \ln\kappa_1 \\ &= -\ln\tau - (s-3)\ln((s-3)\alpha_\tau+\beta_\tau+\delta_\tau) - \ln\left(\frac{\delta_\tau((s-3)\alpha_\tau+\beta_\tau+\delta_\tau)}{\alpha_\tau^2}\right) \\ &\quad + 2(s-2)\ln\alpha_\tau - (s-1)\ln(rs-r-s-s\alpha_\tau) + \ln\kappa_4 + \ln\kappa_1. \end{aligned}$$

We will use primes to denote differentiation with respect to τ .

Lemma A.2. *Suppose that the assumptions of Lemma 5.6 hold. The functions*

$$\alpha_\tau + \beta_\tau + \delta_\tau, \quad \frac{\delta_\tau((s-3)\alpha_\tau + \beta_\tau + \delta_\tau)}{\alpha_\tau^2}$$

are both strictly increasing with respect to $\tau \in [1, \infty)$.

Proof. Recalling that p_τ and q_τ both increase with τ , it follows that

$$\frac{\delta_\tau}{\alpha_\tau} = \left(1 + p_\tau/q_\tau + \frac{(s-3)}{2\kappa_4} + \sqrt{\frac{(s-2)(1+p_\tau/q_\tau)}{\kappa_4} + \frac{(s-3)^2}{4\kappa_4^2}} \right)^{-1}$$

and

$$\frac{\delta_\tau((s-3)\alpha_\tau + \beta_\tau + \delta_\tau)}{\alpha_\tau^2} = \kappa_4 \left(1 - \frac{1}{1 + \frac{(s-3)}{2\kappa_4(1+p_\tau/q_\tau)} + \sqrt{\frac{(s-2)}{\kappa_4(1+p_\tau/q_\tau)} + \frac{(s-3)^2}{4\kappa_4^2(1+p_\tau/q_\tau)^2}}} \right)^2$$

are both increasing functions of τ . Hence, by (A.5), it follows that δ_τ increases. Finally, observe that $p_\tau < 1$ for all $\tau \geq 1$, so

$$\frac{d}{d\tau}(\alpha_\tau + \beta_\tau + \delta_\tau) = \alpha'_\tau - \alpha'_\tau p_\tau + (1 - \alpha_\tau)p'_\tau + \delta'_\tau > 0,$$

completing the proof. \square

Using Lemma A.2 and the fact that $\beta_\tau + \delta_\tau \leq \alpha_\tau \leq 1$, we calculate that

$$\begin{aligned} \varphi'_\alpha(\mathbf{x}_\tau) &\leq -\frac{1}{\tau} - (s-3) \frac{(s-3)\alpha'_\tau + \beta'_\tau + \delta'_\tau}{(s-3)\alpha_\tau + \beta_\tau + \delta_\tau} + 2(s-2) \frac{\alpha'_\tau}{\alpha_\tau} + \frac{(s-1)s\alpha'_\tau}{rs - r - s - s\alpha_\tau} \\ &\leq -\frac{1}{\tau} + \left(2(s-2) - \frac{(s-3)(s-4)}{s-2} + \frac{(s-1)s}{rs - r - 2s} \right) \frac{\alpha'_\tau}{\alpha_\tau}. \end{aligned} \quad (\text{A.10})$$

Observe that, by Lemma A.1,

$$\varphi'(\mathbf{x}_\tau) = \varphi_\alpha(\mathbf{x}_\tau) \alpha'_\tau. \quad (\text{A.11})$$

Using (A.5), it follows that $\varphi'(\mathbf{x}_\tau)$ and $\varphi_\alpha(\mathbf{x}_\tau)$ always have the same sign. Hence, the next lemma implies that the function $\tau \mapsto \varphi(\mathbf{x}_\tau)$ is concave when τ is large enough. We prove Lemma A.3 in Section A.2 below.

Lemma A.3. *Suppose that $s \geq 3$ and $r > \rho(s)$. Then $\varphi'_\alpha(\mathbf{x}_\tau) < 0$ whenever one of the following conditions holds:*

- (i) $\tau \geq 2(s+2)^2$,
- (ii) $(r-1)(s-1) \geq (s+1)^3$ and $\tau \geq (s+1)^{2.5}$.

The rest of the argument is split into two cases.

Case 1: r is sufficiently large

Assume that $(r-1)(s-1) \geq (s+1)^3$. Since $\varphi_\alpha(\mathbf{x}^*) = 0$, it follows from Lemma A.3(ii) that

$$\varphi_\alpha(\mathbf{x}_\tau) > 0 \text{ for } (s+1)^{2.5} \leq \tau < \tau^* \quad \text{and} \quad \varphi_\alpha(\mathbf{x}_\tau) < 0 \text{ for } \tau > \tau^*. \quad (\text{A.12})$$

By (A.11), this shows that $\varphi(\mathbf{x}_\tau)$ is a concave function of τ when $\tau \geq (s+1)^3$. So there can be at most one local maximum of $\varphi(\mathbf{x}_\tau)$ in $[(s+1)^3, \infty)$, and we know that this local maximum occurs at $\tau = \tau^*$. Combining this with Lemma A.1 and (A.9), we conclude in particular that there are no global maxima of φ on the boundary of K .

To complete the proof in this case, we will show that $\varphi(\mathbf{x}_\tau) < \varphi(\mathbf{x}^*)$ for any $\mathbf{x}_\tau \neq \mathbf{x}^*$ such that $\varphi_\alpha(\mathbf{x}_\tau) = 0$ and $\tau \neq \tau^*$. That is, we may restrict our attention to values of τ which correspond to stationary points of φ . For a contradiction, suppose that $\varphi(\mathbf{x}_{\tilde{\tau}}) \geq \varphi(\mathbf{x}^*)$ for some $\tilde{\tau} > 1$ with $\tilde{\tau} \neq \tau^*$ and $\varphi_\alpha(\mathbf{x}_{\tilde{\tau}}) = 0$. Then $1 < \tilde{\tau} < (s+1)^{2.5}$, by (A.12). We solve the equations $\frac{\partial \varphi}{\partial \alpha}(\mathbf{x}_{\tilde{\tau}}) = \frac{\partial \varphi}{\partial \beta}(\mathbf{x}_{\tilde{\tau}}) = \frac{\partial \varphi}{\partial \gamma}(\mathbf{x}_{\tilde{\tau}}) = \frac{\partial \varphi}{\partial \delta}(\mathbf{x}_{\tilde{\tau}}) = 0$ for $\ln \kappa_1, \ln \kappa_2, \ln \kappa_3, \ln \kappa_4$, and then substitute these expressions into (5.12), to obtain

$$\varphi(\mathbf{x}_{\tilde{\tau}}) = \ln \tilde{\tau} + \frac{s-1}{s}(rs - r - s) \ln(rs - r - s - s\tilde{\alpha})$$

where $\tilde{\alpha} = \alpha_{\tilde{\tau}}$. By the lower bound on r we have

$$r \geq \frac{(s+1)^3}{s-1} + 1 \geq 33 \quad \text{and} \quad \frac{rs - r - s}{rs - r - 2s} \leq 1 + \frac{s}{(s+1)^3 - s - 1} \leq 1.05. \quad (\text{A.13})$$

Then

$$\begin{aligned} 0 \leq \varphi(\mathbf{x}_{\tilde{\tau}}) - \varphi(\mathbf{x}_1) &= \ln \tilde{\tau} + \frac{s-1}{s}(rs - r - s) \ln \left(1 - \frac{s\tilde{\alpha}}{rs - r - s} \right) \\ &\leq \ln \tilde{\tau} - (s-1)\tilde{\alpha} \end{aligned} \quad (\text{A.14})$$

and

$$\begin{aligned} 0 \leq \varphi(\mathbf{x}_{\tilde{\tau}}) - \varphi(\mathbf{x}^*) &= \ln \tilde{\tau} - \ln \tau^* + \frac{s-1}{s}(rs - r - s) \ln \left(\frac{rs - r - s - s\tilde{\alpha}}{rs - r - s - s\alpha^*} \right) \\ &\leq \ln \tilde{\tau} - 3 \ln(s+1) + \frac{s-1}{s}(rs - r - s) \ln \left(1 + \frac{s(1-\tilde{\alpha})}{rs - r - 2s} \right) \\ &\leq \ln \tilde{\tau} - 3 \ln(s+1) + (s-1) \frac{rs - r - s}{rs - r - 2s} (1 - \tilde{\alpha}) \\ &\leq \ln \tilde{\tau} - 3 \ln(s+1) + 1.05(s-1)(1 - \tilde{\alpha}), \end{aligned} \quad (\text{A.15})$$

using (A.13) for the final inequality. Taking a carefully chosen linear combination of the inequalities (A.14), (A.15), we conclude that

$$0 \leq 1.05(1 - \tilde{\alpha})(\varphi(\mathbf{x}_{\tilde{\tau}}) - \varphi(\mathbf{x}_1)) + \tilde{\alpha}(\varphi(\mathbf{x}_{\tilde{\tau}}) - \varphi(\mathbf{x}_1))$$

$$\leq 1.05 \ln \tilde{\tau} - 3\tilde{\alpha} \ln(s+1). \quad (\text{A.16})$$

Since $\tilde{\tau} < (s+1)^{2.5}$, this implies that

$$\tilde{\alpha} \leq \frac{1.05 \ln(\tilde{\tau})}{3 \ln(s+1)} < \frac{1.05 \times 2.5}{3} = 0.875. \quad (\text{A.17})$$

Now observe that by (A.8),

$$1 - (1 - \tilde{\alpha})^{-1} + p_{\tilde{\tau}} + q_{\tilde{\tau}} \leq 0.$$

In Lemma A.4, stated later, we will prove that

$$1 - (1 - \tilde{\alpha})^{-1} + p_{\tilde{\tau}} + q_{\tilde{\tau}} \geq T(\tilde{\alpha}) \quad (\text{A.18})$$

where

$$T(\alpha) = -(1 - \alpha)^{-1} + 1 + \frac{(50^\alpha - 1)^2 + 2(50^\alpha - 1)}{(50^\alpha - 1)^2 + 4(50^\alpha - 1) + 2} + R(\alpha)$$

and

$$R(\alpha) = \begin{cases} 0 & \text{if } \alpha < 0.35, \\ \frac{15}{62} 50^\alpha \frac{(50^\alpha - 1)^2}{(50^\alpha - 1)^2 + 4(50^\alpha - 1) + 2} & \text{otherwise.} \end{cases}$$

From the plot of the function $\alpha \mapsto T(\alpha)$ given in Figure 3, we observe that $T(\alpha)$ is strictly positive for $0 < \alpha \leq 0.875$. Therefore, using (A.17), if (A.18) holds then

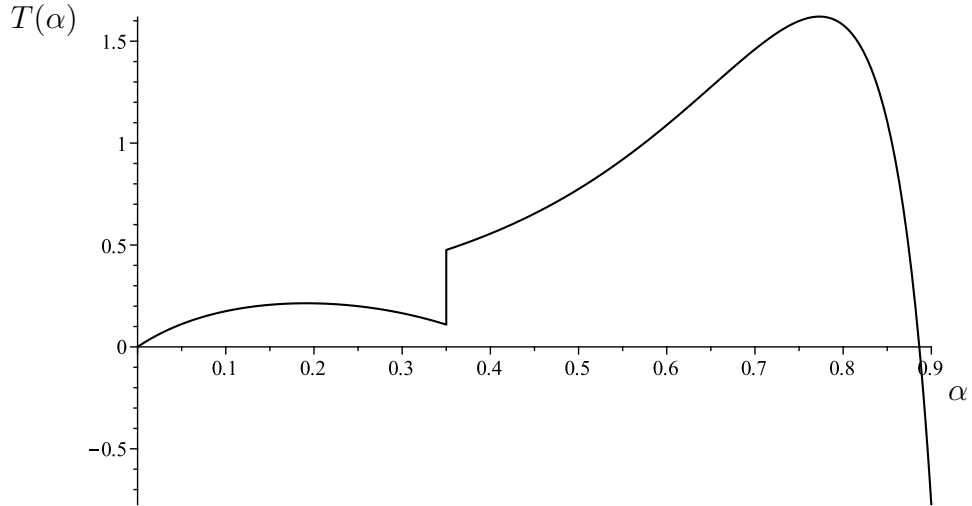


Figure 3: The function $T(\alpha)$, which is positive for $0 < \alpha \leq 0.875$.

$$1 - (1 - \tilde{\alpha})^{-1} + p_{\tilde{\tau}} + q_{\tilde{\tau}} > 0.$$

But this contradicts (A.8). Therefore no other local maximum $\tilde{\tau}$ of $\varphi(\mathbf{x}_\tau)$ can exist in the interval $[1, (s+1)^{2.5}]$.

To complete the proof of the lemma in the case that $(r-1)(s-1) \geq (s+1)^3$, we must establish (A.18).

Lemma A.4. *If $s \geq 3$ and $(r-1)(s-1) \geq (s+1)^3$ then (A.18) holds.*

Proof. Recalling the definition of κ_2 in (5.7), and since

$$r-2 \geq \frac{(s+1)^3}{s-1} - 1 \geq (s+2)^2,$$

it follows from

$$\kappa_2 = s^2 - s + \frac{2(s-2)(s-1)}{r-2} + \frac{(s-2)(s-3)}{(r-2)^2} \quad (\text{A.19})$$

$$\begin{aligned} &\leq s^2 - s + \frac{2(s-1)(s-2)}{(s+2)^2} + \frac{(s-2)(s-3)}{(s+2)^4} \\ &\leq s^2 - 1. \end{aligned} \quad (\text{A.20})$$

By (A.16), since $s \geq 3$ we have

$$\tilde{\tau} \geq (s+1)^{\frac{3}{1.05}\tilde{\alpha}} \geq 50^{\tilde{\alpha}},$$

while if $\tilde{\alpha} \geq 0.35$ then we can instead write

$$\tilde{\tau} \geq (s+1)4^{\frac{3}{1.05}\tilde{\alpha}-1} \geq \frac{s+1}{4} 50^{\tilde{\alpha}}.$$

Noting that $\kappa_3 \leq 2$, we can estimate

$$p_{\tilde{\tau}} \geq \frac{(50^{\tilde{\alpha}} - 1)^2 + 2(50^{\tilde{\alpha}} - 1)}{(50^{\tilde{\alpha}} - 1)^2 + 4(50^{\tilde{\alpha}} - 1) + 2}$$

and if $\tilde{\alpha} \geq 0.35$ we have

$$q_{\tilde{\tau}} \geq \frac{\kappa_4(s+1)}{4\kappa_2} 50^{\tilde{\alpha}} \frac{(50^{\tilde{\alpha}} - 1)^2}{(50^{\tilde{\alpha}} - 1)^2 + 4(50^{\tilde{\alpha}} - 1) + 2}.$$

Additionally, it follows from (A.13) that $\kappa_4 = \frac{r-3}{r-2} \geq \frac{30}{31}$. Using these inequalities, together with the upper bound on κ_2 given in (A.20), we conclude that (A.18) holds, as required. \square

Case 2: r is small

It remains to consider the case that $(r-1)(s-1) < (s+1)^3$. By definition of $\rho(s)$ (see Table 1), the only remaining pairs (r, s) belong to the set

$$\mathcal{A} = \{(r, 3) \mid r = 4, \dots, 32\} \cup \{(r, 4) \mid r = 6, \dots, 41\} \cup \{(r, 5) \mid r = 12, \dots, 54\} \\ \cup \{(r, 6) \mid r = 28, \dots, 69\} \cup \{(r, 7) \mid r = 65, \dots, 86\}.$$

There are 172 pairs (r, s) in \mathcal{A} . (It may be possible to reduce this number by refining the above analysis, but we have not pursued this.) For $(r, s) \in \mathcal{A}$, we argue as follows.

For $(r, s) \in \mathcal{A}$ we consider two functions. The first is

$$\tau \mapsto \tau^{-(s-4)/(s-2)} (\varphi(\mathbf{x}^*) - \varphi(\mathbf{x}_\tau))$$

on the interval $\tau \in [1, s+1]$, and the second is

$$\tau \mapsto -\frac{\tau}{\ln(\tau)} \varphi'_\alpha(\mathbf{x}_\tau)$$

on the interval $\tau \in [s+1, 2(s+2)^2]$. Figures 4–8 show the plots of these functions for all $(r, s) \in \mathcal{A}$, with all pairs with a given value of s displayed together.

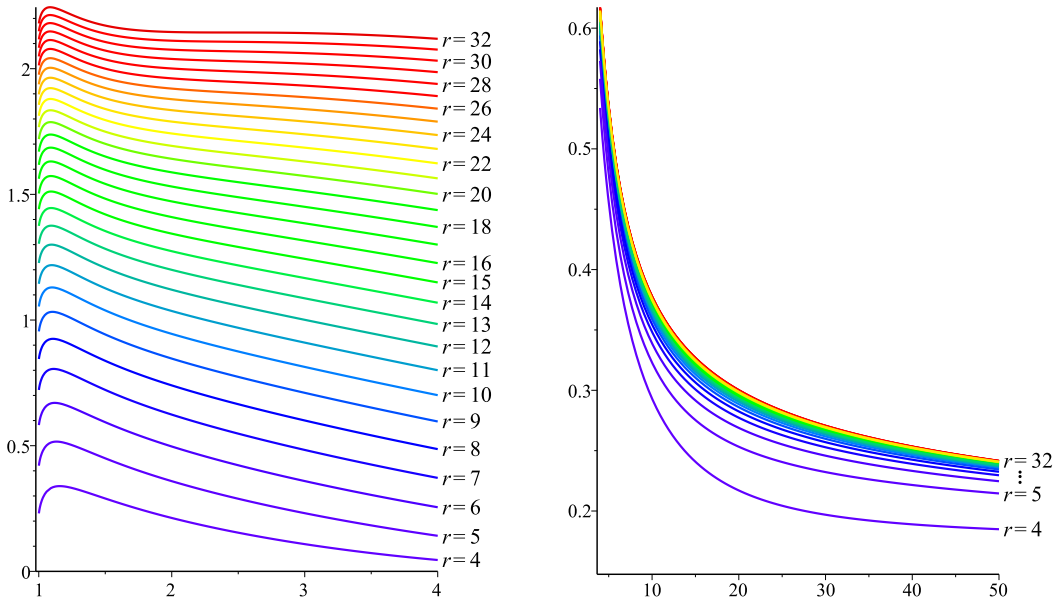


Figure 4: The two plots for $s = 3$, $r = 4, \dots, 32$.

In each of these figures, the plot of the first function is shown on the left, and the plot of the second function is shown on the right. The colours are used to help distinguish between the plots for different values of r . (The scaling factors $\tau^{-(s-4)/(s-2)}$ and $\tau/\ln(\tau)$)

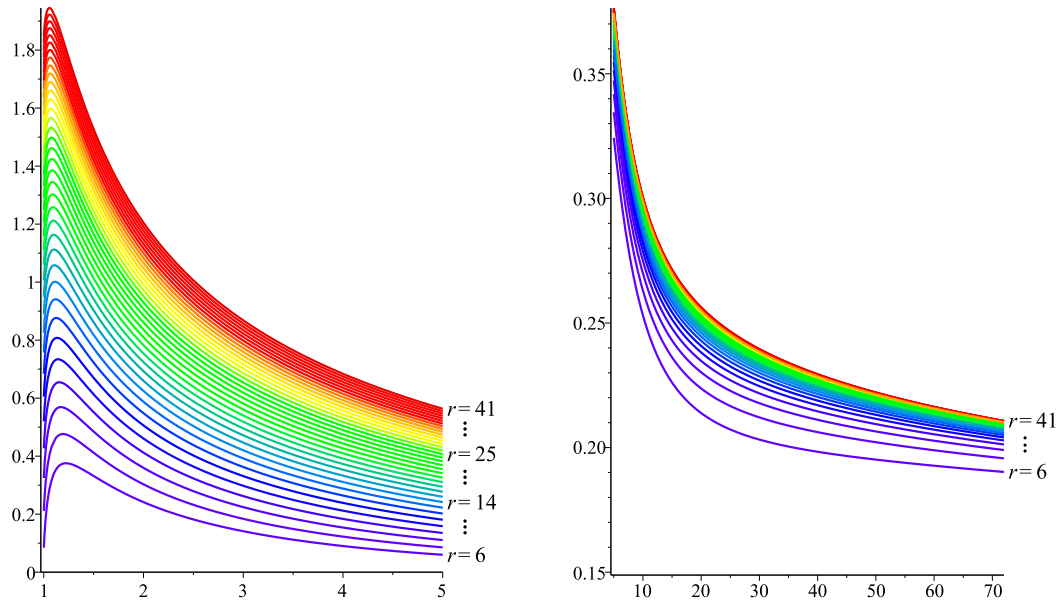


Figure 5: The two plots for $s = 4$, $r = 6, \dots, 41$.

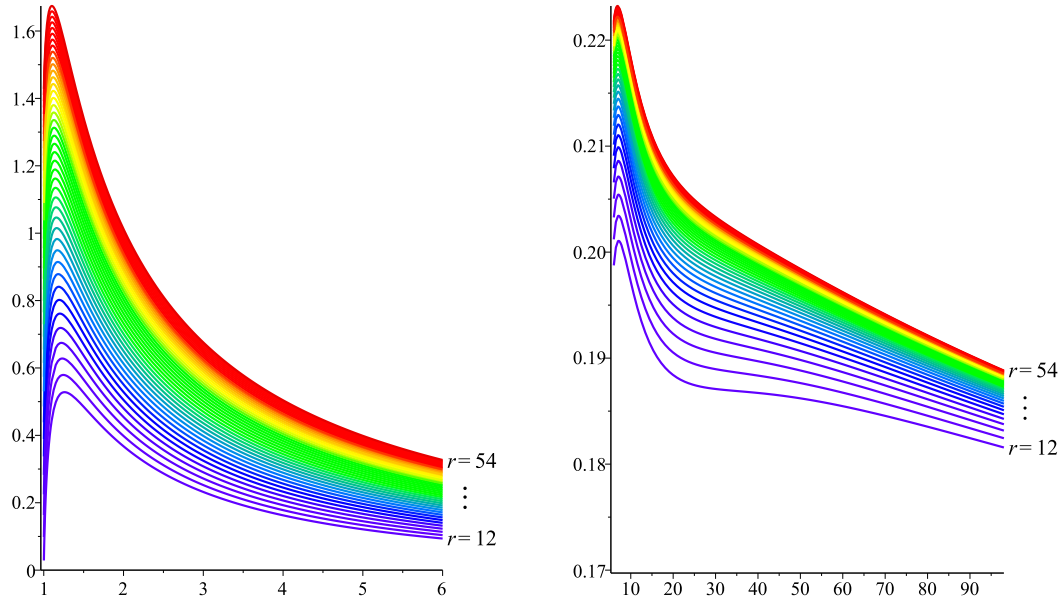


Figure 6: The two plots for $s = 5$, $r = 12, \dots, 54$.

in the first and second plot, respectively, do not affect the sign of the functions, and are included to attempt to spread out the different plots shown in each figure.)

Consider each $(r, s) \in \mathcal{A}$ in turn: we can see that both plots are strictly positive over

the given intervals. The first plot (on the left) shows that $\varphi(\mathbf{x}_\tau)$ is strictly less than $\varphi(\mathbf{x}^*)$ for all $\tau \in [1, s + 1]$.

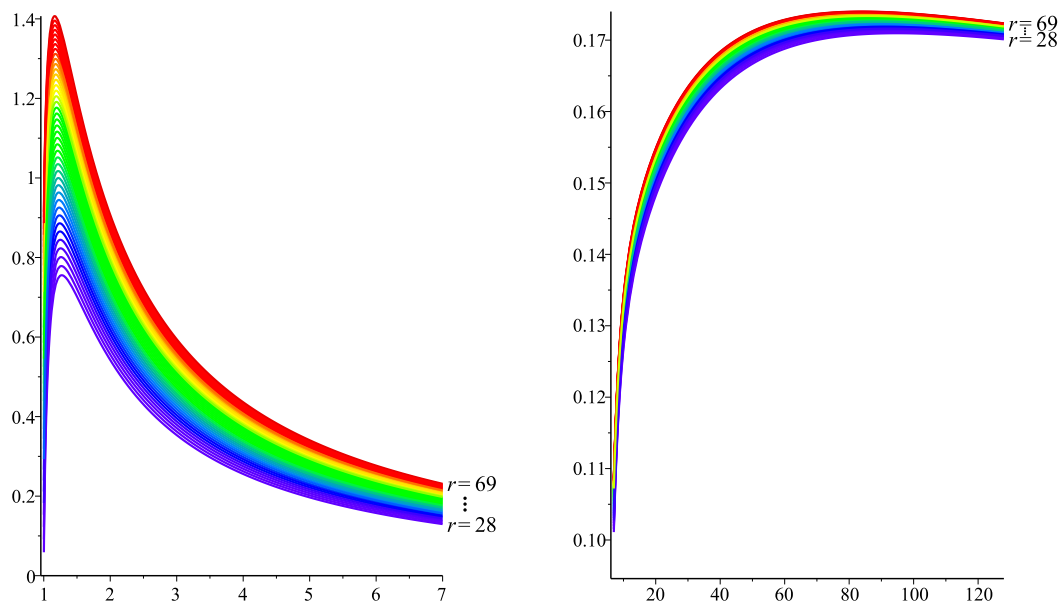


Figure 7: The two plots for $s = 6$, $r = 28, \dots, 69$.

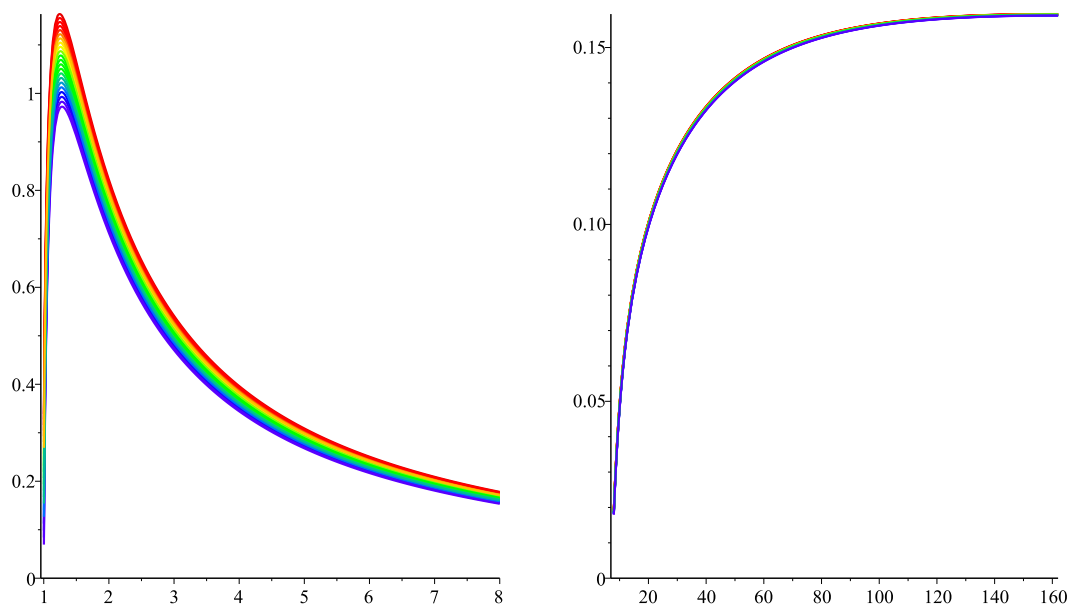


Figure 8: The two plots for $s = 7$, $r = 65, \dots, 86$.

Combining the second plot (on the right) with Lemma A.3(i), using (A.11), we conclude that the function $\tau \mapsto \varphi(\mathbf{x}_\tau)$ is concave for $\tau \geq s + 1$. Therefore $\varphi(\mathbf{x}_\tau)$ has at most one local maximum in the interval $[s + 1, \infty)$.

But we know that $\mathbf{x}^* = \mathbf{x}_{\tau^*}$ is a local maximum of $\varphi(\mathbf{x}_\tau)$, and $\tau^* = (r - 1)(s - 1) > s + 1$ when $s \geq 3$ and $r \geq s + 1$. Combining all this, we conclude that \mathbf{x}^* is the unique global maximum of $\varphi(\mathbf{x}_\tau)$ on $[1, \infty)$.

This argument covers the 172 remaining cases of $(r, s) \in \mathcal{A}$ and completes the proof of Lemma 5.6.

A.1 Proof of Lemma A.1

Since η_τ is continuous and K_τ is compact, the function η_τ attains its maximum at least once on K_τ . For $\tau = 1$ the region K_τ consist of one point $(\beta_\tau, \gamma_\tau, \delta_\tau)$, so the lemma is true for this case. In the following we assume that $\tau > 1$, which implies that $0 < \alpha_\tau < 1$. Let

$$G_{\tau,\beta}(\gamma) = \frac{\gamma^2}{(\beta - \gamma)(1 - \alpha_\tau - \beta - \gamma)}, \quad D_{\tau,\beta}(\delta) = \frac{((s - 3)\alpha_\tau + \beta + \delta)\delta}{(\alpha_\tau - \beta - \delta)^2}.$$

Note that $G_{\tau,\beta}(\gamma) = \kappa_3$ if and only if $\frac{\partial \varphi}{\partial \gamma}(\alpha_\tau, \beta, \gamma, \delta) = 0$, by (A.3). For any $0 < \beta < \min\{\alpha_\tau, 1 - \alpha_\tau\}$, the function $G_{\tau,\beta}(\gamma)$ is strictly increasing with respect to γ and

$$G_{\tau,\beta}(0) = 0, \quad \lim_{\gamma \rightarrow \min\{\beta, 1 - \alpha_\tau - \beta\}} G_{\tau,\beta}(\gamma) = \infty.$$

Therefore, there is a unique value of γ , say $\gamma = \tilde{\gamma}_\tau(\beta)$, which satisfies $G_{\tau,\beta}(\gamma) = \kappa_3$. Note that $\tilde{\gamma}_\tau(\beta) < \beta$ for all $\beta \in (0, \min\{\alpha_\tau, 1 - \alpha_\tau\})$, since κ_3 is finite. To cover cases $\beta = 0$ and $\beta = 1 - \alpha_\tau$, we continuously extend $\tilde{\gamma}_\tau$ and put $\tilde{\gamma}_\tau(0) = \tilde{\gamma}_\tau(1 - \alpha_\tau) = 0$.

Similarly $D_{\tau,\beta}(\delta) = \kappa_4$ if and only if $\frac{\partial \varphi}{\partial \delta}(\alpha_\tau, \beta, \gamma, \delta) = 0$, by (A.4). For any $0 \leq \beta < \min\{\alpha_\tau, 1 - \alpha_\tau\}$, the function $D_{\tau,\beta}(\delta)$ is strictly increasing with respect to δ and

$$D_{\tau,\beta}(0) = 0, \quad \lim_{\delta \rightarrow \alpha_\tau - \beta} D_{\tau,\beta}(\delta) = \infty.$$

Therefore, there is a unique value of δ , say $\delta = \tilde{\delta}_\tau(\beta)$, such that $D_{\tau,\beta}(\delta) = \kappa_4$. Note that $\tilde{\delta}_\tau(\beta) < \alpha_\tau - \beta$ for all $\beta \in [0, \min\{\alpha_\tau, 1 - \alpha_\tau\})$, since κ_4 is finite. We continuously extend $\tilde{\delta}_\tau$ by defining $\tilde{\delta}_\tau(\alpha_\tau) = 0$.

We now show that no local maximum of η_τ can lie on a boundary of K_τ . Suppose that $(\beta_\times, \gamma_\times, \delta_\times)$ is a local maximum of η_τ in K_τ .

- (i) If $\beta_\times = 0$ then $\gamma_\times = 0$ and $\delta_\times = \tilde{\delta}_\tau(0) \in (0, \alpha_\tau)$. (The fact that $\tilde{\delta}_\tau(0) < \alpha_\tau$ arises by definition of $\tilde{\delta}_\tau(0)$, since κ_4 is finite.) From (A.2), we find that, for β in the neighbourhood of 0,

$$\frac{\partial \eta_\tau}{\partial \beta}(\beta, \gamma_\times, \delta_\times) = -\ln \beta + O(1) > 0.$$

Therefore, the point $(\beta_\times, \gamma_\times, \delta_\times)$ cannot be a local maximum of $\eta_\tau(\beta, \gamma_\times, \delta_\times)$ if $\beta_\times = 0$.

- (ii) If $\beta_{\times} = \alpha_{\tau} < 1 - \alpha_{\tau}$ then $\delta_{\times} = 0$ and $\gamma_{\times} = \tilde{\gamma}_{\tau}(\alpha_{\tau}) \in (0, \alpha_{\tau})$. From (A.2), we find that, for β in the neighbourhood of α_{τ} ,

$$\frac{\partial \eta_{\tau}}{\partial \beta}(\beta, \gamma_{\times}, \delta_{\times}) = 2 \ln(\alpha_{\tau} - \beta) + O(1) < 0.$$

Therefore, the point $(\beta_{\times}, \gamma_{\times}, \delta_{\times})$ cannot be a local maximum of $\eta_{\tau}(\beta, \gamma_{\times}, \delta_{\times})$ if $\beta_{\times} = \alpha_{\tau}$ and $0 < \alpha_{\tau} < 1/2$.

- (iii) If $\beta_{\times} = 1 - \alpha_{\tau} \leq \alpha_{\tau}$ then $\gamma_{\times} = 0$ and $\delta_{\times} = \tilde{\delta}_{\tau}(1 - \alpha_{\tau})$. From (A.2), (A.3) and $\frac{\partial \varphi}{\partial \gamma} = 0$, we find that for β in the neighbourhood of $1 - \alpha_{\tau}$,

$$\begin{aligned} \frac{\partial \eta_{\tau}}{\partial \beta}(\beta, \tilde{\gamma}_{\tau}(\beta), \delta_{\times}) &= -\ln \left(\frac{1 - \alpha_{\tau} - \beta}{1 - \alpha_{\tau} - \beta - \tilde{\gamma}_{\tau}(\beta)} \right) + 2 \ln(\alpha_{\tau} - \beta - \delta_{\times}) + O(1) \\ &\leq -\ln \left(1 + \frac{\beta - \tilde{\gamma}_{\tau}(\beta)}{\kappa_3 \tilde{\gamma}_{\tau}(\beta)} \right) + O(1) \\ &= -\ln \left(1 + \frac{\kappa_3(\beta - \tilde{\gamma}_{\tau}(\beta))}{\tilde{\gamma}_{\tau}(\beta)} \right) + O(1) < 0. \end{aligned}$$

(We only need to take care with the term $2 \ln(\alpha_{\tau} - \beta - \delta_{\times})$ when $\alpha_{\tau} = 1/2$, in which case $\delta_{\times} = 0$. But since this term is negative we may bound it above by zero.) Therefore, the point $(\beta_{\times}, \gamma_{\times}, \delta_{\times})$ cannot be a local maximum of $\eta_{\tau}(\beta, \tilde{\gamma}_{\tau}(\beta), \delta_{\times})$ if $\beta_{\times} = 1 - \alpha_{\tau}$ and $\frac{1}{2} \leq \alpha_{\tau} < 1$.

Thus, the point $(\beta_{\times}, \gamma_{\times}, \delta_{\times})$ lies in the interior of the domain K_{τ} , and hence it satisfies the system of equations $\frac{\partial \eta_{\tau}}{\partial \beta} = \frac{\partial \eta_{\tau}}{\partial \gamma} = \frac{\partial \eta_{\tau}}{\partial \delta} = 0$. Define $\tau_{\times} = \frac{1 - \alpha_{\tau} - \beta_{\times}}{1 - \alpha_{\tau} - \beta_{\times} - \gamma_{\times}}$. From $\frac{\partial \eta_{\tau}}{\partial \gamma} = 0$ we find that

$$\tau_{\times} - 1 = \frac{\gamma_{\times}}{1 - \alpha_{\tau} - \beta_{\times} - \gamma_{\times}} = \frac{\kappa_3(\beta_{\times} - \gamma_{\times})}{\gamma_{\times}},$$

which gives us $\beta_{\times} = (\frac{\tau_{\times} - 1}{\kappa_3} + 1)\gamma_{\times}$. Substituting this back into the expression for τ_{\times} and solving with respect to γ_{\times} , we obtain

$$\gamma_{\times} = (1 - \alpha_{\tau}) \frac{\tau_{\times} - 1}{\frac{(\tau_{\times} - 1)^2}{\kappa_3} + 2\tau_{\times} - 1}.$$

Next, since $\frac{\partial \eta_{\tau}}{\partial \beta} = 0$ and $\frac{\partial \eta_{\tau}}{\partial \delta} = 0$, we have

$$\delta_{\times} = \frac{\kappa_4(\beta_{\times} - \gamma_{\times})(1 - \alpha_{\tau} - \beta_{\times})}{\kappa_2(1 - \alpha_{\tau} - \beta_{\times} - \gamma_{\times})} = \frac{\kappa_4}{\kappa_2 \kappa_3} \tau_{\times} (\tau_{\times} - 1) \gamma_{\times} = (1 - \alpha_{\tau}) \frac{\kappa_4 \tau_{\times} (\tau_{\times} - 1)^2}{\kappa_2 \kappa_3 \left(\frac{(\tau_{\times} - 1)^2}{\kappa_3} + 2\tau_{\times} - 1 \right)}.$$

Recalling the definitions of p_{τ} and q_{τ} , we can write

$$\beta_{\times} = (1 - \alpha_{\tau}) p_{\tau_{\times}}, \quad \delta_{\times} = (1 - \alpha_{\tau}) q_{\tau_{\times}}.$$

Substitute these expressions into the equation $\frac{\partial \eta_\tau}{\partial \delta} = 0$ to obtain

$$(1 - \alpha_\tau)^{-1} = 1 + p_{\tau_\times} + q_{\tau_\times} + \frac{(s-3)q_{\tau_\times}}{2\kappa_4} \pm \sqrt{\frac{(s-2)q_{\tau_\times}(p_{\tau_\times} + q_{\tau_\times})}{\kappa_4} + \frac{(s-3)^2 q_{\tau_\times}^2}{4\kappa_4^2}}.$$

Since $\beta_\times + \delta_\times \leq \alpha_\tau$, which is equivalent to $(1 - \alpha_\tau)^{-1} \geq 1 + p_{\tau_\times} + q_{\tau_\times}$, we must take the positive sign outside of the radical. Therefore, it follows that

$$(1 - \alpha_\tau)^{-1} = (1 - \alpha_{\tau_\times})^{-1}.$$

By monotonicity of α_τ we conclude that $\tau_\times = \tau$, and hence $(\beta_\times, \gamma_\times, \delta_\times) = (\beta_\tau, \gamma_\tau, \delta_\tau)$. Therefore, the point $(\beta_\tau, \gamma_\tau, \delta_\tau)$ is the only point where the global maximum of η_τ is attained on K_τ . This completes the proof of Lemma A.1.

A.2 Proof of Lemma A.3

We will use (A.10). Observe that the factor multiplying $\frac{\alpha'_\tau}{\alpha_\tau}$ in (A.10) can be bounded above by

$$2s - 4 - \frac{(s-3)(s-4)}{s-2} + \frac{s(s-1)}{rs - r - 2s} \leq s + 2 \quad (\text{A.21})$$

when $r \geq s + 1 \geq 4$.

We now work towards an upper bound on $\frac{\alpha'_\tau}{\alpha_\tau}$. Write $\alpha_\tau = \frac{N_\tau}{1+N_\tau}$ where

$$N_\tau = p_\tau + q_\tau + \frac{s-3}{2\kappa_4}q_\tau + \sqrt{\frac{s-2}{\kappa_4}q_\tau(p_\tau + q_\tau) + \frac{(s-3)^2}{4\kappa_4^2}q_\tau^2}.$$

Then

$$\frac{\alpha'_\tau}{\alpha_\tau} = \frac{N'_\tau}{N_\tau(1+N_\tau)}. \quad (\text{A.22})$$

It is not difficult to check that $1 \leq \kappa_3 \leq 2$, which implies that

$$1 - \frac{2}{\tau} \leq p_\tau \leq 1, \quad q_\tau \geq \frac{\kappa_4}{\kappa_2}(\tau - 4),$$

$$\begin{aligned} p'_\tau &= \frac{\kappa_3(\tau^2 + \kappa_3 - 1)}{(\tau^2 + (\kappa_3 - 1)(2\tau - 1))^2} \leq \frac{\kappa_3}{\tau^2 + (\kappa_3 - 1)(2\tau - 1)} \leq \frac{2}{\tau^2}, \\ q'_\tau &= \frac{\kappa_4(\tau - 1)((\tau - 1)^3 + \kappa_3(4\tau^2 - 3\tau + 1))}{\kappa_2((\tau - 1)^2 + \kappa_3(2\tau - 1))^2} \leq \frac{\kappa_4}{\kappa_2}. \end{aligned}$$

Recall that $\mu = 2\kappa_4 + s - 3$. Applying (A.7) and the above inequalities, we have

$$N_\tau \geq p_\tau + 2q_\tau + \frac{s-3}{\kappa_4}q_\tau \geq 1 - \frac{2}{\tau} + \frac{\mu(\tau - 4)}{\kappa_2} = \frac{\mu\tau}{\kappa_2} \left(1 + \frac{\kappa_2}{\mu\tau} - \frac{4}{\tau} - \frac{2\kappa_2}{\mu\tau^2} \right)$$

and

$$\begin{aligned}
N'_\tau &\leq p'_\tau + q'_\tau + \frac{s-3}{2\kappa_4} q'_\tau + \frac{\frac{s-2}{\kappa_4} (q_\tau p'_\tau + q'_\tau p_\tau + 2q_\tau q'_\tau) + \frac{(s-3)^2}{2\kappa_4^2} q_\tau q'_\tau}{\mu q_\tau} \\
&= p'_\tau + 2q'_\tau + \frac{s-3}{\kappa_4} q'_\tau + \frac{(s-2)p'_\tau + 2(1-\kappa_4)q'_\tau}{\mu} + \frac{(s-2)q'_\tau p_\tau}{\mu q_\tau} \\
&\leq \frac{2}{\tau^2} + \frac{\mu}{\kappa_2} + \frac{2(s-2)}{\mu\tau^2} + \frac{2\kappa_4(1-\kappa_4)}{\kappa_2\mu} + \frac{s-2}{\mu(\tau-4)} \\
&= \frac{\mu}{\kappa_2} \left(1 + \frac{2\kappa_4(1-\kappa_4)}{\mu^2} + \frac{(s-2)\kappa_2}{\mu^2(\tau-4)} + \frac{2\kappa_2}{\mu\tau^2} + \frac{2(s-2)\kappa_2}{\mu^2\tau^2} \right).
\end{aligned}$$

Substituting these bounds into (A.22), we conclude that

$$\frac{\alpha'_\tau}{\alpha_\tau} \leq \frac{\frac{\kappa_2}{\mu\tau^2} \left(1 + \frac{2\kappa_4(1-\kappa_4)}{\mu^2} + \frac{(s-2)\kappa_2}{\mu^2(\tau-4)} + \frac{2\kappa_2}{\mu\tau^2} + \frac{2(s-2)\kappa_2}{\mu^2\tau^2} \right)}{\left(1 + \frac{\kappa_2}{\mu\tau} - \frac{4}{\tau} - \frac{2\kappa_2}{\mu\tau^2} \right) \left(1 + \frac{2\kappa_2}{\mu\tau} - \frac{4}{\tau} - \frac{2\kappa_2}{\mu\tau^2} \right)}. \quad (\text{A.23})$$

Using (5.7), we claim that whenever $s \geq 3$ and $r > \rho(s)$, we have

$$\begin{aligned}
\frac{1}{2} &\leq \kappa_4 \leq 1 \quad \text{and} \quad \kappa_4(1-\kappa_4) \leq \frac{1}{2}, \\
s-2 &\leq \mu \leq s-1, \\
s^2-s &\leq \kappa_2 \leq s^2+s-3, \\
s &\leq \frac{\kappa_2}{\mu} \leq \begin{cases} 8 & \text{if } s=3, \\ s+3 & \text{if } s \geq 4. \end{cases}
\end{aligned}$$

For almost all these inequalities, the weak bound $r \geq s+1$ is sufficient. The bounds on κ_4 are clear, and lead immediately to the bounds on μ . Use (A.19) for the bounds on κ_2 , with $r-2 \geq s-1$. The lower bound on κ_2/μ then follows, while for the upper bound we must be a little more precise. If $s \geq 5$ then we bound $r-2 \geq s-1$ in the definition of κ_2 , giving

$$\frac{\kappa_2}{\mu} \leq \frac{s^2+s-3}{s-1-\frac{2}{s-1}} \leq s+3.$$

If $s=4$ then we have $r > \rho(4) > 5$, so $\mu \geq \frac{5}{2}$ and

$$\frac{\kappa_2}{\mu} \leq \frac{34}{5} < 7 = s+3.$$

Finally, if $s=3$ then $r \geq 4$ and $\mu \geq 1$, while (A.19) implies that $\kappa_2 \leq 8$.

First suppose that $s \geq 4$. We prove (i) and (ii) at the same time (and indeed, prove that the condition on (r, s) in (ii) is unnecessary when $s \geq 4$) by showing that

$\varphi'_\alpha(\mathbf{x}_\tau) < 0$ whenever $\tau \geq \min\{2(s+2)^2, (s+1)^{2.5}\}$. Using the inequalities proved above, the denominator of (A.23) is bounded below by

$$\begin{aligned} & \left(1 + \frac{\kappa_2}{\mu\tau} - \frac{4}{\tau} - \frac{2\kappa_2}{\mu\tau^2}\right) \left(1 + \frac{2\kappa_2}{\mu\tau} - \frac{4}{\tau} - \frac{2\kappa_2}{\mu\tau^2}\right) \\ & \geq \left(1 + \frac{s-4}{\tau} - \frac{2s}{\tau^2}\right) \left(1 + \frac{2(s-2)}{\tau} - \frac{2s}{\tau^2}\right) \\ & \geq 1 + \frac{3s-8}{\tau} + \frac{2(s^2-8s+8)}{\tau^2} - \frac{2s(3s-8)}{\tau^3}. \end{aligned}$$

If $s \geq 7$ then $s^2 - 8s + 8 \geq 0$ and the denominator of (A.23) is bounded below by 1, since $\tau^2 - 2s \geq 0$ whenever $\tau \geq \min\{2(s+2)^2, (s+1)^{2.5}\}$. For $s = 4, 5, 6$, direct substitution into the above expression confirms that the denominator of (A.23) is bounded below by 1.

Using this, we may apply our inequalities to the numerator of (A.23) to obtain

$$\frac{\alpha'_\tau}{\alpha_\tau} \leq \frac{s+3}{\tau^2} \left(1 + \frac{1}{2(s-2)^2} + \frac{s+3}{\tau-4} + \frac{4(s+3)}{\tau^2}\right).$$

Substituting this and (A.21) into (A.10), we find that $\varphi'_\alpha(\mathbf{x}_\tau) < 0$ if

$$(s+2)(s+3) \left(1 - \frac{1}{2(s-2)^2} + \frac{s+3}{\tau-4} + \frac{4(s+3)}{\tau^2}\right) < \tau. \quad (\text{A.24})$$

If $s \geq 5$ then the left hand side of (A.24) is bounded above by

$$(s+2)(s+3) \left(1 + \frac{1}{18} + \frac{8}{84} + \frac{32}{88^2}\right) = \frac{8804}{7623}(s+2)(s+3),$$

which is bounded above by $\min\{2(s+2)^2, (s+1)^{2.5}\}$. Hence (i) and (ii) hold whenever $s \geq 5$ and $r > \rho(s)$.

When $s = 4$ we have $r > \rho(4) > 5$ and $\min\{2(s+2)^2, (s+1)^{2.5}\} = 5^{2.5} > 55$. We bound the left hand side of (A.21) by $26/5$. Substituting this into (A.21), we see that when $\tau \geq 55$, the left hand side of (A.24) is bounded above by

$$\frac{7 \times 26}{4} \left(1 + \frac{1}{8} + \frac{7}{51} + \frac{28}{55}\right) < 55.$$

This implies that (i) and (ii) hold whenever $s = 4$ and $r > \rho(4)$.

For the final case, suppose that $s = 3$. Then we must use the bound $\frac{\kappa_4}{\mu} \leq 8$, so the numerator of (A.23) becomes

$$8 \left(\frac{3}{2} + \frac{8}{\tau-4} + \frac{32}{\tau^2}\right). \quad (\text{A.25})$$

For (ii) we also assume that $2(r-1) \geq 4^3$, which means that $r \geq 33$. Hence the left hand side of (A.21) is bounded above by $\frac{21}{10}$, and for all $\tau \geq 32$ the left hand side of (A.24) is bounded above by

$$8 \times \frac{21}{10} \left(\frac{3}{2} + \frac{8}{28} + \frac{1}{32}\right)$$

which is strictly less than 32, as required. This proves that (ii) holds when $s = 3$.

To prove (i) when $s = 3$, assume that $r \geq 4$. The left hand side of (A.21) is bounded above by $2 + \frac{3}{r-3}$, and the numerator of (A.23) is bounded above by the expression given in (A.25). Therefore a sufficient condition for $\varphi'_\alpha(\mathbf{x}_\tau)$, when $s = 3$, is

$$8 \left(2 + \frac{3}{r-3} \right) \left(\frac{3}{2} + \frac{8}{\tau-4} + \frac{32}{\tau^2} \right) < \tau.$$

If $r \geq 5$ then this inequality holds for all $\tau \geq 50$. Therefore (i) holds when $s = 3$ and $r \geq 5$.

Finally, suppose that $s = 3$ and $r = 4$. Then

$$\kappa_4 = \frac{1}{2}, \quad \mu = 1, \quad \kappa_2 = 8.$$

Substituting these values into (A.10), we find that $\varphi'_\alpha(\mathbf{x}_\tau) < 0$ if

$$60 \left(1 + \frac{4}{\tau-4} + \frac{16}{\tau^2} \right) < \tau \left(1 + \frac{16}{\tau} + \frac{16}{\tau^2} - \frac{256}{\tau^3} \right).$$

This inequality holds whenever $\tau \geq 50$, completing the proof that (i) holds whenever $s = 3$ and $r = 4$. This completes the proof of Lemma A.3. \square